WHAT RESTS ON WHAT?
THE PROOF-THEORETIC ANALYSIS OF MATHEMATICS

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Whenever a subject is organized systematically for expository or foundational purposes (or both), one must deal with the question: What rests on what? The way in which this is answered in the case of mathematics depends on whether one is considering it informally or formally, i.e. from the point of view of the mathematician or the logician, respectively. The latter usually deals with the question in terms of what specifically follows from what in a given logical/axiomatic setup. Proof theory provides technical notions and results which–when successful–serve to give a more global kind of answer to this question, in terms of reduction of one such system to another; moreover, these results provide a technical bridge from mathematics to philosophy. The purpose of this paper is to give a picture of what is accomplished in these various respects by reductive proof theory. My own approach to that subject is outlined in §1, along with a brief comparison with a more standard account. This is then followed in §2 by a description of some technical results that illustrate the general approach. The paper concludes in §3 with a discussion of how reductive proof theory mediates between mathematics and philosophy. A more technical and comprehensive exposition of the material in §§1-2 has previously been given in my paper [Feferman 1988].

§1. What rests on what? Proof-theoretical and foundational reductions.

In the following we use the letters:

$\mathcal{M}$, for an informal part of mathematics (such as number theory, analysis, algebra, etc., or a subdivision of such);

L, for a formal language for a part of mathematics (e.g. the language of elementary number theory);

$\phi, \psi, \ldots$ for well-formed formulas or statements of L;

T, for a formal axiomatic system in L (e.g. the system of first-order Peano Arithmetic PA in the language of elementary number theory); and

$\mathcal{F}$, for a general foundational framework (e.g. finitary, constructive, predicative, countable infinitary, set-theoretical or uncountable infinitary, etc.).

These categories provide different senses in which we can deal with the question of what rests on what from a logical point of view:

$\mathcal{M}$ *rests on* T, in the sense that $\mathcal{M}$ *can be formalized in* T;
ϕ *rests on* T, in the sense that ϕ *is provable in* T;
T *rests on* $\mathcal{F}$, in the sense that T *is justified by* $\mathcal{F}$; and
T_1 *rests on* T_2, in the sense that T_1 *is reducible to* T_2.

With respect to the last of these, there are different technical notions of reducibility of one axiomatic system to another. We want to contrast, in particular, the notion of T_1 being *interpretable (or translatable)* in T_2 with that of T_1 being *proof-theoretically reducible*

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to T_2, written $T_1 \leq T_2$ (this will be defined in §2). In general, these move in opposite directions from a foundational point of view, since we are mainly concerned with the relation $T_1 \leq T_2$ when T_2 is a *part* of T_1, either directly or by translation. In contrast, T_2 tends to be more comprehensive than T_1 in the case of interpretations; a familiar example is that of Peano Arithmetic PA (as T_1) in Zermelo-Fraenkel set theory ZF (as T_2), where the natural numbers are interpreted as the finite ordinals. This is a *conceptual reduction* of number theory to set theory, but not a *foundational reduction*, because the latter system is justified only by an uncountable infinitary framework whereas the former is justified simply by a countable infinitary framework.

The driving aim of the original Hilbert program (H.P.) was to provide a finitary justification for the use of the "actual infinite" in mathematics. This was to be accomplished by directly formalizing one body or another of infinitary mathematics $\mathcal{M}$ in a formal axiomatic theory T_1 and then demonstrating the consistency of T_1 by purely finitary means; in practice, that would be established by a proof-theoretic reduction of T_1 to a system T_2 justified on finitary grounds. It is generally acknowledged that H.P. as originally conceived could not be carried through even for elementary number theory PA as the system T_1, in consequence of Gödel's 1931 incompleteness theorems. This then gave rise to certain relativized forms of H.P.; the history will be traced briefly below. In our approach, what the results of a relativized H.P. should achieve are best expressed in the following way. ([Feferman 1988], p. 364):

> A body of mathematics $\mathcal{M}$ is represented directly in a formal system T_1 which is justified by a foundational framework $\mathcal{F}_1$. T_1 is reduced proof-theoretically to a system T_2 which is justified by another, more elementary such framework $\mathcal{F}_2$.

In Hilbert's scheme, $\mathcal{F}_1$ was to be the infinitary framework of modern mathematics featuring (i) the "completed" or "actual" infinite (both countable and uncountable) and (ii) non-constructive reasoning, while $\mathcal{F}_2$ was to be the framework of finitary mathematics featuring (i)′ only the "potential" infinite of finite combinatorial objects, and (ii)′ constructive reasoning applied to quantifier-free statements (typically, equations). According to Hilbert, already the system PA embodies (i) and (ii) by the use of quantified variables which are supposed to range over the set N of natural numbers and the assumption of the Law of the Excluded Middle, from which follows statements of the form $(\forall x)\phi(x) \vee (\exists x)\neg\phi(x)$. Even for decidable quantifier-free ϕ these require for their justification a survey of the totality of natural numbers, by a kind of infinitary act of omniscience. [2]

The general problem raised by Gödel's incompleteness results [1931] for H.P. is that if finitary mathematics is itself to count as a significant body of informal mathematics, it must be formalizable in a consistent formal axiomatic theory T. Then by Gödel's second incompleteness theorem, the consistency of T would not be provable in T, hence could not be finitarily provable, and so H.P. cannot be carried out for T. Just what T could serve this

[2] This view in the Hilbert school is revealed, for example, by the title of [Ackermann 1924–25]: "Begründung des 'tertium non datur' mittels der Hilbertschen Theorie des Widerspruchsfreiheit", and by Hilbert's reference to the axioms for quantification as the "transfinite axioms".

purpose was not analyzed in the Hilbert school. In fact, the kind of finitary reasoning that had been employed by its workers in the 1920s could evidently be formalized in PA and even in its quantifier-free fragment PRA (Primitive Recursive Arithmetic), and it would be hard to imagine anything at all of the same character which could not be formalized in Zermelo set theory or the equivalent finite theory of types P of [Gödel 1931].[3] Nevertheless, Gödel was cautious at the time about the significance of his second incompleteness theorem for H.P.: "I wish to note expressly that [this theorem does] not contradict Hilbert's formalistic viewpoint . . . it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism of P . . . ".[4] And, in his foreword to [Hilbert and Bernays 1934], Hilbert also (implicitly) stressed this possibility, saying that the only consequence of Gödel's incompleteness theorems for his program was that one would have to work harder than anticipated in order to carry it out.

Despite Hilbert's continued optimism, the general feeling after 1931 was that Gödel's second incompleteness theorem doomed H.P. to failure, and that some essentially new idea would be needed to carry out anything like it, even for PA. Yet another result of Gödel in his paper [1933] (independently found by Bernays and Gentzen) forced a further reconsideration of H.P.: This showed that PA could be translated in a simple way into the intuitionistic system HA of Heyting's arithmetic, which differs from PA only in omitting the Law of the Excluded Middle from its basic logical principles. Thus, one specific goal of H.P. was achieved in a single stroke (cf. ftn. 2 above). Moreover, the intuitionists argued that HA is justified on a conception of the natural numbers as a potentially infinite set. From that point of view, the result of Gödel's translation of PA into HA was to eliminate the actual infinite in favor of the potential infinite and thus to meet the main aim of H.P. at least for number theory. However, by its free use of the language of first-order quantification theory, the system HA is not thereby justified on the basis of the more strictly finitary forms of reasoning using only quantifier-free formulas, as demanded by Hilbert. On the one hand, Gödel's 1933 result corrected a common tendency in the Hilbert school prior to 1933 to identify intuitionism with finitism. On the other hand, his reduction of PA to HA fueled the growing search for a wider conception of H.P. As reported by Bernays ([1967], p. 502) years later: "It thus became apparent that the 'finite Standpunkt' is not the only alternative to classical ways of reasoning and is not necessarily implied by the idea of proof theory. An enlarging of the methods of proof theory was therefore suggested: instead of a reduction to finitist methods of reasoning, it was required only that the arguments be of a constructive character, allowing us to deal with more general forms of inference." Here was the germ of a relativized form of H.P.

One strikingly new specific way forward was provided by Gentzen in his paper [1936] in which the consistency of PA was proved by transfinite induction up to Cantor's ordinal

[3] The question of completely formalizing finitary mathematics has been addressed by [Kreisel 1958] and [Tait 1981]; the former arrived at a system of the same strength as PA, while the latter restricted it much more drastically to PRA.

[4] Cf. [Gödel 1986], pp. 138-139 and 195. Gödel later sharply revised this opinion in his paper [1958]: "[I]t is necessary to go beyond the framework of what is, in Hilbert's sense, finitary mathematics if one wants to prove the consistency of classical mathematics, or even that of classical number theory." (Cf. [Gödel 1990], p. 241). There is evidence that he had arrived at that point of view long before 1958.

ϵ_0 $(\mathrm{TI}(\epsilon_0))$ applied only to decidable quantifier-free predicates and otherwise using only finitary reasoning. Here the ordinals less than ϵ_0 are represented in Cantor normal form $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ $(\alpha_1 \geq \ldots \geq \alpha_n)$, and the ordering of these is isomorphic to an effective ordering of the natural numbers by taking the sequence number $p_1^{a_1} \ldots p_n^{a_n}$ to correspond to $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ when a_i corresponds to α_i. The principle $\mathrm{TI}(\epsilon_0)$ may itself be justified on constructive grounds (though no longer clearly finitary grounds, even for decidable predicates). Gentzen showed that his result was best possible by establishing each instance of $\mathrm{TI}(\beta)$ in PA for each $\beta < \epsilon_0$. In modern terms, ϵ_0 was thus identified as the "ordinal of PA", in the sense of being the least non-provably recursive well-ordering in PA. Gentzen's consistency proof apparently impelled Bernays' acceptance of a further shift away from the original H.P., as is evidenced by its inclusion in [Hilbert and Bernays 1939], under the section title: "Überschreitung des bisherige methodischen Standpunktes der Beweistheorie".[5]

In the post-war period, Gentzen-style consistency proofs and ordinal analysis of various subsystems of analysis and set theory became the dominant approach in proof theory. These have involved the construction of more and more complicated recursive well-orderings $\prec$ of the natural numbers, obtained from notation systems for larger and larger ordinals α, in each case identified as the least non-provably recursive well-ordering of the theory T in question; furthermore, the consistency of T is proved by transfinite induction up to $\alpha, \mathrm{TI}(\alpha)$ (i.e. on the corresponding recursive $\prec$ relation) applied to decidable predicates, while otherwise using only finitary reasoning. Several modern texts which exposit this kind of extension of the Gentzen approach are: [Schütte 1977], [Takeuti 1987] and [Girard 1987], as well as the monograph [Pohlers 1990]. These texts are highly technical, and cannot be faulted on mathematical grounds; on the contrary, they contain many deep results. But it is not at all clear what they contribute to an extended H.P. in the sense envisioned by Bernays. The crucial question is: In what sense is the assumption of $\mathrm{TI}(\alpha)$ justified constructively for the very large ordinals α used in these consistency proofs? Indeed, on the face of it, the explanation of which ordinals α are used appeals to the very concepts and results of infinitary set theory that one is trying to account for on constructive grounds. For example, in Ch. IX of [Schütte 1977], a notation system is introduced which is defined in terms of a hierarchy of normal functions on the set of ordinals less than the first fixed point of $\aleph_\alpha = \alpha$. But this is only the beginning. In the still more advanced research of [Jäger and Pohlers 1982], use is made of a notation system based on the ordinals less than the first (set-theoretically) inaccessible cardinal, to establish the consistency of a moderately strong subsystem of analysis. And [Rathjen 1991] has used notation systems based on the ordinals less than the first Mahlo cardinal to prove the consistency of a subsystem of set theory. The notation systems developed for these purposes are all countable, though they name extremely large uncountable ordinals. Then, by means of an analysis of the ordering relations, one shows in each case that the ordering of the notations is recursive. Moreover, well-ordering proofs of a more or less constructive character can be given which do not appeal to the fact that the notation systems are derived from hierarchies of functions on very large ordinal number classes. However, the conviction that one is indeed dealing

[5] The preparation of [Hilbert and Bernays 1934 and 1939] had been placed by Hilbert entirely in the hands of Bernays.

with well-ordering relations derives from the latter and not from the well-ordering proofs in which those traces have been obliterated. Finally, there is a *prima facie* anomaly in the use of transfinite induction applied to orderings for enormous ordinals while insisting on a restriction to finitary reasoning otherwise. Having given up the original H.P. in favor of a reduction of classical infinitary mathematics to one part or another of constructive mathematics, there seems to be no reason to retain its finitary vestiges.

One more feature of the extended Hilbert-Gentzen program deserves criticism, namely the focus on consistency proofs: Hardly anyone nowadays doubts that Zermelo-Fraenkel set theory is consistent or, to be much more moderate, that the system of analysis (full second-order arithmetic) is consistent. And surely the number who doubt that PA is consistent is vanishingly small. It is true that Takeuti ([1987], p. 100–101) and Schütte ([1977], p. 3) still pay lip service to the goal of consistency proofs, but their followers have de-emphasized that in favor of other goals, such as the ordinal analysis of formal systems (cf. [Pohlers 1990], pp. 5-6).

A more thoroughgoing reconsideration of H.P. was initiated by Kreisel in his paper "Mathematical significance of consistency proofs"([1958a]) and "Hilbert's programme" ([1958b]). The former emphasized the additional mathematical information that can be gained from successful applications of proof-theoretical methods, by telling what more we know of a statement, beyond its truth, if we know that it has been proved by specific methods. For example, this may take the form of extracting bounds for existential results, or for the complexity of provably recursive functions. The latter paper first suggested the idea of a "hierarchy" of Hilbert programs; this was elaborated in [Kreisel 1968], which considered besides reductions to finitary and constructive conceptions also reductions to semi-constructive (e.g. predicative) conceptions, and within each a more refined analysis of what principles are needed for various pieces of reductive work (op. cit. pp. 323-324).

My own approach in [Feferman 1988] to a relativized form of H.P. formulated as $(*)$ above, is similar in this over-all respect to [Kreisel 1968] but (to repeat myself, op. cit. p. 367): "the details are different both as to the categorization of the conceptions to which the foundational reductions are referred and as to the proof-theoretical work which exemplifies these reductions. Concerning the latter, it is simply that most of the work surveyed has been carried out in the last twenty years. And, with respect to the former, we have tried to seize on *the most obvious features of foundational conceptions* [or *frameworks*] so that, *insofar as possible, what the work achieves will speak for itself."*

Some examples from my 1988 survey which illustrate the scheme $(*)$ are given in the next section.

§2. Hilbert's program relativized: Proof-theoretical and foundational reductions.

2.1. Proof-theoretical reductions. All systems T considered in the following are assumed to contain PRA (described in 2.3 below). The language $\mathrm{L_T}$ of T and the axioms and rules of inference of T are assumed to be specified by primitive recursive presentations via usual Gödel numberings (coding) of expressions; we may identify expressions with their codes. Thus the relation $Proof_{\mathrm{T}}(x, y)$ which holds when y is (the code of) a proof in T of the formula (with code) x, is primitive recursive. The metatheory of primitively recursively presented axiomatic systems can be formalized directly in PA and even in a subsystem of PA which is a conservative extension of PRA (described in 2.4 below); for details, cf. [Smoryński 1977] or [Feferman 1989].

When considering a pair of systems T_1, T_2, we write L_i for $\mathrm{L}_{\mathrm{T}_i}$ (i = 1,2). Suppose Φ is a primitive recursive class of formulas contained in both L_1 and L_2, which contains every closed equation $\mathrm{t}_1 = \mathrm{t}_2$. The basic idea of a proof-theoretic reduction of T_1 to T_2 conserving Φ is that we have an effective method of transforming each proof in T_1 ending in a formula ϕ of Φ into a proof of ϕ in T_2; moreover, we should be able to establish that transformation provably within T_2. More precisely, we say that T_1 is *proof-theoretically reducible to* T_2 *by* f, *conservatively for* Φ, and write

$$f : \mathrm{T}_1 \leq \mathrm{T}_2 \text{ for } \Phi$$

if f is a partial recursive function such that

(1) whenever $Proof_{\mathrm{T}_1}(x, y)$ and x is (the code of) a formula in Φ then $f(y)$ is defined and $Proof_{\mathrm{T}_2}(x, f(y))$,

and

(2) the formalization of (1) is provable in T_2.

We write $\mathrm{T}_1 \leq \mathrm{T}_2$ for Φ, if there is such an f satisfying (1) and (2). In practice, f may be chosen to be primitive recursive and the formalization of (1) can be proved in PRA.

It is immediate that if $\mathrm{T}_1 \leq \mathrm{T}_2$ for Φ, then T_1 *is conservative over* T_2 *for* Φ in the sense that

(3) $\phi \in \Phi$ and $\mathrm{T}_1 \vdash \phi$ implies $\mathrm{T}_2 \vdash \phi$.

It then follows by the general assumption on Φ that

(4) if T_2 is consistent then T_1 is consistent,

since if $\mathrm{T}_1 \vdash 0 = 1$ then $\mathrm{T}_2 \vdash 0 = 1$. Moreover, by (2), the formalization $\mathrm{Con}_{\mathrm{T}_2} \rightarrow \mathrm{Con}_{\mathrm{T}_1}$ of (4) is provable in T_2 (and, in practice, already in PRA.)

Remark. It should be noted that we may have T_1 conservative over T_2 without there being any possible proof-theoretic reduction of T_1 to T_2. For example, if Φ is the class of closed equations $\mathrm{t}_1 = \mathrm{t}_2$ of the language of PRA and T_1 is any consistent extension of PRA then

T_1 is conservative over PRA for Φ, because if $T_1 \vdash t_1 = t_2$ we must have PRA $\vdash t_1 = t_2$, for otherwise PRA $\vdash t_1 \neq t_2$. Now choose T_1 to be any consistent system which proves Con_{PRA}, so it is not proof-theoretically reducible to PRA (e.g., to be extreme, T_1 = ZF).

2.2. Foundational reductions. According to our general scheme($*$) of §1, a proof-theoretical reduction $T_1 \leq T_2$ provides a *partial foundational reduction* of a framework $\mathcal{F}_1$ to $\mathcal{F}_2$, if T_1 is justified directly by $\mathcal{F}_1$ and T_2 by $\mathcal{F}_2$. The reason for the qualification 'partial' is that we may well have a system T_1 which is directly justified by $\mathcal{F}_1$ but which is not reducible to *any* T_2' justified by $\mathcal{F}_2$. For example (to be extreme again), the system ZF, which is justified by the uncountable infinitary framework of Cantorian set theory, is not reducible to any finitarily justified system. On the other hand, it may also happen that a system T_1 which *prima-facie* requires for its justification an appeal to an infinitary framework is proof-theoretically reducible to a finitarily justified T_2; in that case we have a partial reduction of the infinitary to the finitary.

In the sections 2.4 and 2.6-2.8 below we shall describe some results which exemplify partial foundational reductions for the following pairs of frameworks:

$\mathcal{F}_1$	$\mathcal{F}_2$
Infinitary	Finitary
Uncountable Infinitary	Countable Infinitary
Impredicative	Predicative
Non-constructive	Constructive

Remarks.
(i) The pairs $\mathcal{F}_1$, $\mathcal{F}_2$ are not the only ones which might be considered in this respect (cf. [Feferman 1988] p. 367).
(ii) In establishing proof-theoretical reductions which provide partial foundational reductions we may well use results of Gentzen-Schütte-Takeuti style, but these appear behind the scenes. The point is to apply such work to results which speak for themselves (unlike consistency proofs by transfinite induction on very large ordinals).
(iii) The emphasis here on the use of proof theory for a form of relativized H.P. is not meant to diminish other applications of proof theory – on the contrary. But the interest of such is guided by quite different considerations.

2.3. The language and basic axioms of first-order arithmetic. In order to describe the reductive results for several systems of arithmetic in the next section, we need some syntactic and logical preliminaries. The language L_0 is a type 0 (or first-order) single-sorted formalism. It contains variables x, y, z, ..., the constant symbol 0, the successor symbol $'$ and symbols $\mathbf{f}_0, \mathbf{f}_1, \ldots$ for each primitive recursive function, beginning with $\mathbf{f}_0(x,y)$ for $x + y$ and $\mathbf{f}_1(x,y)$ for $x \cdot y$. Terms $t, t_1, t_2, \ldots$ are built up from the variables and 0 by closing under $'$ and the $\mathbf{f}_i$. The atomic formulas are equations $t_1 = t_2$ between terms. Formulas are built up from these by closing under the propositional operations ($\neg, \wedge, \vee, \rightarrow$) and quantification ($\forall$ x, $\exists$ x) with respect to any variable. A formula is said to be quantifer-free, or in the class QF, if it contains no quantifiers. A formula is said to be in the class Σ^0_n if it is given by n alternating quantifiers beginning with '$\exists$', followed by a QF matrix. For example, $(\exists \mathrm{y})\psi \in \Sigma^0_1$ and $(\exists \mathrm{y}_1)(\forall \mathrm{y}_2)\psi \in \Sigma^0_2$ when $\psi \in$ QF. (The superscript in 'Σ^0_n' indicates that these are type 0 variables, and the 'Σ' tells us that the quantifier string starts with '$\exists$'.) Dually, ϕ is in the class Π^0_n if it is given by n alternating quantifiers beginning with '$\forall$', followed by a QF matrix. For example, $(\forall y)\psi \in \Pi^0_1$ and $(\forall \mathrm{y}_1)(\exists \mathrm{y}_2)\psi \in \Pi^0_2$ when $\psi \in$ QF.

Unless otherwise noted the underlying logic is that of the classical first-order predicate calculus with equality. The basic non-logical axioms Ax_0 of arithmetic are:

(1) $\mathrm{x}' \neq 0$,
(2) $\mathrm{x}' = \mathrm{y}' \to \mathrm{x} = \mathrm{y}$,
(3) $\mathrm{x} + 0 = \mathrm{x} \wedge \mathrm{x} + \mathrm{y}' = (\mathrm{x} + \mathrm{y})'$,
(4) $\mathrm{x} \cdot 0 = 0 \wedge \mathrm{x} \cdot \mathrm{y}' = \mathrm{x} \cdot \mathrm{y} + \mathrm{x}$,

and so on for each further $\mathbf{f}_i$ (using its primitive recursive defining equations as axioms). To Ax_0 may be added certain instances of the *Induction Axiom* scheme:

$$\text{IA} \qquad \phi(0) \wedge \forall \mathrm{x}(\phi(\mathrm{x}) \to \phi(\mathrm{x}')) \to \forall \mathrm{x}\phi(\mathrm{x}),$$

for each formula $\phi(\mathrm{x})$ (with possible additional free variables). Φ-IA is used to denote both this scheme restricted to $\phi \in \Phi$, *and* the system with axioms Ax_0 plus Φ-IA. We use PA (Peano Arithmetic) to denote the system with axioms Ax_0 + IA where the full scheme is used.[6]

The language of PRA (Primitive Recursive Arithmetic) is just the quantifier-free part of L_0. In place of IA it uses an Induction Rule:

$$\text{IR} \qquad \frac{\phi(0),\ \phi(\mathrm{x}) \to \phi(\mathrm{x}')}{\phi(\mathrm{x})}, \quad \text{for each } \phi \in \text{QF}.$$

2.4. Reduction of the countable infinitary to the finitary. It is generally acknowledged that PRA is a finitarily justified system, or to be more precise, that each theorem of PRA is finitarily justified.[7] On the other hand, as explained in §1 above, the use of classical quantificational logic in *any* system containing the base axioms (1) $\mathrm{x}' \neq 0$ and (2) $\mathrm{x}' = \mathrm{y}' \to \mathrm{x} = \mathrm{y}$ of Ax_0 implicitly requires assumption of the completed countable infinite. The first result which established a partial reduction of the countable infinitary to finitary principles was that the system QF-IA is proof-theoretically reducible PRA, obtained by Ackermann [1924–25].[8] Years later this was improved by Parsons [1970] to the following:

[6] By a result of [Gödel 1931] all primitive recursive functions are explicitly definable in terms of 0, $'$, $+$, and $\cdot$, and their recursive defining equations are derivable in the subsystem of PA obtained by restriction to formulas in that sublanguage. However, it is more convenient here to take PA in the form described above, so as to include PRA directly.

[7] According to Tait's analysis [1981], finitary number theory coincides with PRA; if that account is accepted, a finitist would recognize each theorem of PRA as being finitarily justified, but not PRA as a whole.

[8] Actually, Ackermann thought he had accomplished much more, namely a consistency proof of analysis by finitary means! His error was discovered by von Neumann [1927] and in essence this relatively modest result was extracted. The situation was further clarified by Herbrand [1930], who gave useful sufficient conditions for finitary consistency proofs.

Theorem 1. Σ^0_1-IA $\leq$ PRA.

The reduction here is conservative for Π^0_2 formulas in the following sense: If Σ^0_1-IA $\vdash \forall$ x $\exists$ y ψ(x,y) with ψ in QF then PRA $\vdash \psi$(x,t(x)) for some term t(x).

Remarks. (i) Here, as below, only references are given in lieu of proofs. (ii) As measured by the arithmetical hierarchy, Theorem 1 is best possible, since the system Σ^0_2-IA proves the consistency of Σ^0_1-IA (cf. [Sieg 1985] pp. 46-47). However, in 2.6 below we shall describe a stronger result obtained by passing to the language of analysis, which is taken up next.

2.5. The language and basic axioms of analysis. The language L_1 of analysis, or second-order arithmetic, is obtained from L_0 by adjunction of variables for type 1 objects. In some presentations, these are taken to be sets, in others, they are taken to be functions, while in still others, both kinds of variables are taken to be basic. For simplicity, we shall follow the first choice here, by adjoining to L_0 set variables X, Y, Z, ... and the binary relation symbol $\in$ between individuals and sets. Thus, the atomic formulas are expanded to include t $\in$ X for any term t and set variable X. (Equality between sets is considered to be defined extensionally, i.e. by: $\mathrm{X} = \mathrm{Y} \leftrightarrow \forall x[x \in \mathrm{X} \leftrightarrow x \in \mathrm{Y}]$.) Formulas are now built up using the propositional operations, and the quantifiers applied both to individual variables ($\forall$x, $\exists$x) and set variables ($\forall$ X, $\exists$X). The underlying logic is now that of classical two-sorted predicate calculus with equality (in the first sort). The classes QF, Σ^0_n and Π^0_n are explained in L_1 just as for L_0 in §2.3 above, but with the understanding that formulas in these classes might contain set variables via the expanded class of atomic formulas. A formula is said to be *arithmetical* if it contains no bound set variables, and we write Arith for the class of all such formulas.[9] In line with the definition of the classifications Σ^0_n and Π^0_n in §2.3, we define classes Σ^1_n and Π^1_n as follows: A formula ϕ of L_1 is said to be in Σ^1_n if it is given by n alternating set quantifiers beginning with '$\exists$', followed by an arithmetical matrix. For example, $(\exists\mathrm{Y})\psi \in \Sigma^1_1$ and $(\exists\mathrm{Y}_1)(\forall\mathrm{Y}_2)\psi \in \Sigma^1_2$ when $\psi \in$ Arith. (Now the superscript in 'Σ^1_n ' tells us that we are measuring the type 1 quantifier complexity of a formula ϕ.) Dually, ϕ is in the class Π^1_n if it is given by n alternating set quantifiers beginning with '$\forall$', followed by an arithmetical matrix. For example, $(\forall\mathrm{Y})\psi \in \Pi^1_1$ and $(\forall\mathrm{Y}_1)(\exists\mathrm{Y}_2)\psi \in \Pi^1_2$ when $\psi \in$ Arith.

The general set existence axiom is given in L_1 by the *Comprehension Axiom* scheme:

CA $$\exists\mathrm{X}\forall\mathrm{x}\ [\mathrm{x} \in \mathrm{X} \leftrightarrow \phi(\mathrm{x})]$$

where ϕ is a formula of L_1 which does not contain the variable X free but may contain free individual and set variables besides x. The Induction Axiom scheme IA of L_1 is of the same form as in L_0, $\phi(0) \wedge \forall\mathrm{x}\ (\phi(\mathrm{x}) \rightarrow \forall\phi(\mathrm{x}')) \rightarrow \forall\mathrm{x}\ \phi(\mathrm{x})$, but where now ϕ may be any formula of L_1. There is another option for the statement of induction in L_1, namely as the single second-order statement:

I_1 $$\forall\mathrm{X}\ [0 \in \mathrm{X} \wedge \forall\mathrm{x}(\mathrm{x} \in \mathrm{X} \rightarrow \mathrm{x}' \in \mathrm{X}) \rightarrow \forall\mathrm{x}\ (\mathrm{x} \in \mathrm{X})].$$

[9] Every formula ϕ of Arith is logically equivalent to a formula in the class $\bigcup_n \Pi^0_n$, also denoted Π^0_∞.

In the presence of full CA, IA is derivable from I_1. The full system of analysis is given by the axioms Ax_0, CA and IA (or equivalently I_1). We shall consider subsystems over Ax_0, based on various combinations Φ-CA and Ψ-IA where Φ, Ψ are classes of L_1 formulas. In particular, two extremes are given special attention in combination with a given Φ-CA, namely adjunction of full IA or adjunction simply of I_1. In the first case the system is denoted Φ-CA and in the second case it is denoted Φ-CA$\restriction$ (for which the notation Φ-CA$_0$ is also used in some presentations). However, in intermediate cases of adjunction Ψ-IA, the system is designated Φ-CA + Ψ-IA.

The formulas Σ^0_1 of L_0 define the recursively enumerable sets in the standard model $(N, 0, ', +, \cdot, \ldots)$. The recursive sets S are exactly those which are recursively enumerable and whose complement $N - S$ is recursively enumerable, i.e. which are definable both by a Σ^0_1 formula and a Π^0_1 formula. In L_1, all this relativizes to the set variables of the formulas. For example, if ϕ(x,X) is in Σ^0_1 and ψ(x,X) is in Π^0_1 in L_1, and $\forall$x(ϕ(x,X) $\leftrightarrow$ ψ (x,X)) holds in the standard model, then {x| ϕ (x,X)} denotes a set recursive in X, and all sets recursive in X are definable in this way. Thus the (relatively) *Recursive Comprehension Axiom* scheme is formulated in L_1 by:

RCA $\quad \forall x[\phi(x) \leftrightarrow \psi(x)] \rightarrow \exists X \forall x[x \in X \leftrightarrow \phi(x)], \quad$ for $\phi \in \Sigma^0_1, \psi \in \Pi^0_1$

(and X not free in ϕ or ψ). RCA is also denoted Δ^0_1-CA. This generalizes in the obvious way to Δ^0_n-CA by taking $\phi \in \Sigma^0_n, \Pi^0_n$. Similarly, we shall consider the scheme

Δ^1_1-CA $\quad \forall x[\phi(x) \leftrightarrow \psi(x)] \rightarrow \exists X \forall x[x \in X \leftrightarrow \phi(x)] \quad$ for $\phi \in \Sigma^1_1, \psi \in \Pi^1_1$

(and X not free in ϕ or ψ) and its obvious generalization to Δ^1_n-CA. The classes Δ^1_1 and Δ^1_2 of sets definable in the standard model are very significant in higher recursion theory. In particular, Δ^1_1 coincides with the class of hyperarithmetic sets, obtained by iterating relative arithmetical (or simply relative Π^0_1) definitions through all recursive ordinals (cf. [Sacks 1990]).

Remark. Second-order arithmetic is called *analysis* because the notion of being a real number can be formally expressed in L_1. First one uses a standard representation of the rational numbers Q in terms of the natural numbers, and then real numbers may be defined as Dedekind sections in Q, i.e. as certain subsets of Q and hence of N. Alternatively, taking functions as the basic type 1 objects, one may formally represent real numbers as Cauchy sequences of rational numbers. To develop a theory of functions of real numbers in general one must proceed to a third-order language L_2; however, many useful classes of functions, such as the continuous functions of reals, may already be handled in L_1. We shall discuss the formalization of informal mathematical analysis in subsystems of CA in §3 below.

2.6. Reduction of the uncountable infinitary to the finitary. Cantor's diagonal argument for the uncountability of the collection of all subsets of the set of natural numbers is easily carried out formally under very minimal assumptions about sets. First of all, a countable sequence of sets $X_0, X_1, \ldots, X_i, \ldots$ is taken to be represented by a single set X for which $y \in X_i \leftrightarrow \langle i, y \rangle \in X$ (where $\langle , \rangle$ is a primitive recursive pairing function). Then it follows from QF-CA that no countable sequence of sets includes all sets, i.e. that $\forall X \exists Y \neg \exists x \forall y [y \in Y \leftrightarrow \langle x, y\rangle \in X]$, by taking $y \in Y \leftrightarrow \langle y, y\rangle \notin X$. Thus if one accepts the Hilbertian point of view that classical quantificational logic over a domain of objects implicitly requires for its justification treating that domain as a completed totality, then assumption of the principle QF-CA (and even much less) requires the uncountable infinitary foundational framework for its justification. Hence if a system T_1 in L_1 includes QF-CA and is proof-theoretically reducible to a system T_2 justified by a finitary framework, then we have a partial reduction of the uncountable infinitary to the finitary. Just such a reduction is given by the following result:

Theorem 2. $$\text{RCA} + \Sigma^0_1\text{-IA} \leq \text{PRA}.$$

The proof of this is motivated by the fact that RCA has an ω-model in the collection of all recursive subsets of N. Formally, one first obtains a direct translation of RCA $+\Sigma^0_1$-IA into Σ^0_1-IA considered as an L_0 system, using the formalization of elementary recursion theory there, and then applies Theorem 1. Once more, the reduction conserves Π^0_2 formulas.

A very interesting strengthening of Theorem 2 has to do with a set existence principle which cannot be realized recursively in this way, namely so-called Weak König's Lemma (WKL). König's Lemma (KL) in general says that any finitely branching infinite tree must contain some infinite path; WKL is its restriction to binary branching trees. The formal statement of WKL in L_1 takes the form:

WKL $$\forall X\ [\text{BinTree}(X) \wedge \text{Inf}(X) \rightarrow \exists Y.\ \text{Path}\ (Y,X)]$$

It is well-known that there are infinite recursive binary trees which contain no recursive paths, so the recursive sets do not form an ω-model of WKL.

Theorem 3. $$\text{RCA} + \text{WKL} + \Sigma^0_1\text{-IA} \leq \text{PRA}.$$

The conservation result which follows from this was first established by Friedman in 1977 by a model-theoretic argument [cf. Simpson 1988]; the proof-theoretical reduction was established in [Sieg 1985].[10,11] This partial reduction of the uncountable infinitary to the finitary is a significant contribution to the original H.P., as has been argued by Simpson [1988].

Remark. Extensions of Theorem 3 which provide partial reductions of the higher uncountable infinitary to the finitary are described in [Feferman 1988], p. 375, using systems of analysis with higher-type function(al) variables.

2.7. Reduction of the uncountable infinitary to the countable infinitary. As was remarked in §2.4 above, Σ^0_2-IA is not proof-theoretically reducible to PRA. Without committing oneself to PRA as the limit of finitistically acceptable principles in arithmetic,

[10]A related earlier result was obtained by [Mints 1976].

[11]Model-theoretic arguments for conservation do not on their face provide proof-theoretical reductions. Friedman has shown that certain forms of model-theoretic arguments can be transformed into such reductions, but this is by no means immediate. Cf. the discussion in [Feferman 1988] pp. 378-381.

it may still be said that we have no evident finitary justification (directly or by reduction) for Σ^0_2-IA or beyond. All instances of the first-order induction scheme as embodied in PA are justified in the countable infinitary framework. We concern ourselves in this subsection with results which establish a partial reduction of the uncountable infinitary to the countable infinitary via results of the form T $\leq$ PA where T is a second-order system. The first of these concerns the system ACA, which is taken to abbreviate Arith-CA. Recall our convention that ACA uses full IA in L_1, while ACA$\restriction$ uses only the restricted induction axiom I_1. ACA$\restriction$ contains PA since every arithmetical formula in L_0 defines a set under Arith-CA.

Theorem 4. $$\text{ACA}\restriction \leq \text{PA}$$

Conservation is for all formulas of L_0. This result is part of the folklore; the earliest reference I'm aware of for a proof (of a more general, but similar) result is [Shoenfield 1954]; cf. also [Feferman and Sieg 1981], p. 112.

Remarks. (i) It is essential here that ACA$\restriction$ uses only the second-order induction axiom I_1 and not the full induction scheme IA in L_1, since ACA (with full IA) proves Con_{PA}. (ii) The arithmetically definable subsets of N form an ω-model of ACA, but by Remark (i) this cannot be used for a translation argument to establish Theorem 4, as it was for Theorem 2. However, there is an easy model-theoretic proof of conservation, since every model M of PA can be expanded to a model of ACA$\restriction$, by taking the sets definable by L_0 formulas from elements of M.

While Theorem 4 may not be unexpected (though the matter is delicate as shown by these remarks) the following *is* unexpected.

Theorem 5. $$\Delta^1_1\text{-CA}\restriction \leq \text{PA}.$$

Here conservation is again for arithmetic formulas. The original proof of the conservation result is due to [Barwise and Schlipf 1975] and, independently [Friedman 1976], in both cases by model-theoretic methods. A proof-theoretical reduction was sketched in [Feferman 1977] p. 967; cf. [Feferman and Sieg 1981a] pp. 108-112 for a full proof.

In the next section we shall consider what can be said about the full system Δ^1_1-CA, which is much stronger than PA.

2.8. Reducing the impredicative to the predicative. A definition of a set S is said to be *predicative* if, roughly speaking, all notions and the ranges of all variables occurring in it are prior to S; otherwise it is called *impredicative* ([cf. Feferman 1964]). Thus, a definition which singles S out from a totality of sets by reference via quantification to that totality is *prima facie* impredicative. In particular, this holds for $S = \{x \mid \phi(x)\}$ where ϕ is a formula of L_1 which contains bound set variables; existence of S is given by CA. Thus Π^1_1-CA (or dually Σ^1_1-CA) is justified only on the assumption of the meaningfulness of impredicative definitions, and that in turn implicitly assumes that there is a well-determined totality of subsets of the natural numbers which exists independently of any means of (human) definition or construction.

An axiomatic characterization of what reasoning with sets of natural numbers is predicatively acceptable has been given independently by [Feferman 1964] and [Schütte 1964]. One way of describing the characterization is that it allows iteration of the system based on relative ACA through all ordinals below a certain recursive ordinal Γ_0, giving a system denoted as $\mathrm{ACA}_{<\Gamma_0}$. It is not hard to show that Π^1_1-CA proves the consistency of $\mathrm{ACA}_{<\Gamma_0}$ (via a proof of the well-foundedness of a notation system for Γ_0), hence cannot be reduced to predicative principles under the above characterization. This leaves unsettled the status of Δ^1_1-CA, which asserts the existence of $\{\mathrm{x}|\ \phi(\mathrm{x})\}$ when ϕ is Π^1_1 and we have $\forall \mathrm{x}(\phi(\mathrm{x}) \leftrightarrow \psi(\mathrm{x}))$ with ψ in Σ^1_1, because that is still *prima facie* impredicative. The following theorem settles the matter, since $\in_0 < \Gamma_0$ and so $\mathrm{ACA}_{<\in_0}$ is a predicatively justified system; it thus constitutes a partial reduction of the impredicative to the predicative.

Theorem 6. $\Delta^1_1\text{-CA} \leq \mathrm{ACA}_{<\in_0}$

The conservation part of this result (for Π^1_2 formulas) was first proved by [Friedman 1970]. A proof-theoretical reduction was outlined in [Feferman 1971] (cf. [Feferman 1977] p. 965), and a full proof by another method has been given in [Feferman and Sieg 1981], pp. 119ff.

2.9. Reducing the non-constructive to the constructive. We use $\mathrm{T}^{(i)}$ to denote the result of substituting intuitionistic logic for classical logic in an axiomatic theory T, otherwise retaining the non-logical axioms and rules of T. (For a suitable formalization of classical logic, this may simply be obtained by dropping the Law of the Excluded Middle.) While intuitionistic logic is deemed to be justified on constructive grounds, this by no means assures that any such $\mathrm{T}^{(i)}$ is constructively acceptable; e.g. no constructive justification for $\mathrm{ZF}^{(i)}$ is known. Gödel's 1933 translation of PA into HA provides an instance of a reduction (by translation) of a system T to $\mathrm{T}^{(i)}$. Since then, a great number of results of this form have been obtained by an extension of Gödel's translation; cf. the introductory note by Troelstra to [Gödel 1933] in [Gödel 1986], pp. 286-287, for a comprehensive survey. Further considerations are necessary in each case to see whether this translation constitutes a reduction of the non-constructive to the constructive. Even so, not all such foundational reductions can be obtained by this relatively simple method. As an example, we have the following result:

Theorem 7. $\Delta^1_2\text{-CA} \leq (\Pi^1_1\text{−CA})^{(i)}_{<\in_0}$.

Here one has conservation for Π^1_2 formulas, as first proved by [Friedman 1970]. The proof-theoretic reduction, which actually establishes more, namely a reduction of Δ^1_2-CA to a constructive theory $\mathrm{ID}^{(i)}_{<\in_0}$ of iterated inductive definitions, was established in a series of steps described in my introduction to [Buchholz et al. 1981], due to work of Pohlers, Sieg and myself. For a survey of further such work, cf. [Feferman 1988], pp. 377-378.

Remark. It happens that Theorems 5–7 can be strengthened directly using forms of the *Axiom of Choice* in L_1:

AC $\qquad \forall \mathrm{x} \exists \mathrm{Y} \phi(\mathrm{x,y}) \rightarrow \exists \mathrm{X} \forall \mathrm{x} \phi(\mathrm{x}, \mathrm{X_x})$.

We have $\Sigma^1_n\text{-AC} \vdash \Delta^1_n\text{-CA}$ by a simple argument. Then 'Δ^1_1-CA' can be replaced by 'Σ^1_1-AC' in Theorems 5 and 6 and 'Δ^1_2-CA' by 'Σ^1_2-AC' in Theorem 7.

2.10. What about full analysis, higher types and set theory? The original H.P. aimed to conquer arithmetic, analysis and set theory, more or less in that order. The examples that have been given in the preceding sections seem to fall far short of giving a foundationally informative reduction of analysis, not to speak of a reduction to finitist principles. Indeed, present day Gentzen-Schütte-Takeuti style proof-theoretical work on subsystems of analysis has failed so far to provide an informative proof-theoretical reduction of Π^1_2 -CA, though not for lack of trying. Whether this is a temporary block is impossible to tell at this time. On the one hand, it is quite possible that the kind of proof-theoretical reduction used here as illustrations of the scheme $(*)$ simply cannot be extended to Π^1_2-CA, let alone to full CA. On the other hand, some new conceptual breakthrough might succeed in dealing with Π^1_2-CA and then open the way to tackle full analysis.

While the future in this direction is completely cloudy, it is possible to say something useful about formal systems for analysis in higher finite types as well as in systems of set theory. For example in [Feferman 1988], I described a system of finite types which includes WKL and is proof-theoretically reducible to PRA (op. cit. p. 375, §4.3); this constitutes a partial reduction of the higher uncountable infinite to the finitary. I also described a system of finite types which includes the system $\Delta^1_1{-}\mathrm{CA}{\restriction}$ and is proof-theoretically reducible to PA (op. cit. p. 375, §5.3); this constitutes a partial reduction of the higher uncountable infinite to the countable infinite. And in [Feferman 1985] I showed how to formulate a flexible theory $\mathrm{VT}_\mu{\restriction}$ of *variable* finite types[12] which also includes $\Delta^1_1{-}\mathrm{CA}{\restriction}$ and is proof-theoretically reducible to PA. This is a theory of functions and classes which is directly interpretable in Zermelo set theory, so the reduction $\mathrm{VT}_\mu{\restriction} \leq \mathrm{PA}$ constitutes a partial reduction of the set-theoretical infinitary to the countable infinitary.

One also obtains a partial reduction of the higher set-theoretical infinitary framework, including transfinite number classes, to the constructive (and in some cases to the predicative framework) through the results of [Jäger 1986]. These provide reductions of various theories of iterated admissible sets, which are contained in ZF, to subsystems of Δ^1_2-CA + BI, where BI is the principle of bar induction.[13] Though these theories lack the Power Set Axiom, which is used in ZF with Replacement to establish the existence of the transfinite (accessible) number classes, they build in the existence of such classes by axioms of iterated admissibility.[14] (Speaking in set-theoretical terms, one has the existence of the alephs, i.e. the cardinals $\aleph_\alpha$ for α accessible in these theories, but not the beths, i.e. the cardinals $\beth_\alpha$ for which $\beth_{\alpha+1} = 2^{\beth_\alpha}$).

It would take us too far afield to try to describe the above systems of higher type, variable type, and set theory in any detail, for which the interested reader should refer to the articles mentioned in this section and in the footnotes.

[12]Denoted Res-VT $+(\mu)$, op. cit. ; cf. alternatively the closely related system W of [Feferman 1988a], §8.

[13]BI expresses that one can carry out transfinite induction on any well-founded orderings; Δ^1_2-CA + BI is weaker than Π^1_2-CA.

[14]This direction of work has been pushed up to Mahlo cardinals by [Rathjen 1991]; what is not evident is whether that gives a reduction to constructive principles.

§3. From mathematics to philosophy via reductive proof theory.[15]

Two matters remain to be dealt with in this final section in order to fill out the scheme $(*)$ of §1. The first is to say something about the passage by formalization from a body of mathematics $\mathcal{M}$ to a formal theory T, especially with reference to the systems presented in §2. The second is to indicate the philosophical significance of the kind of reduction of T_1 to T_2 illustrated by the results in §2. We take these up in that order.

The scheme $(*)$ calls for $\mathcal{M}$ to have a direct formalization in T_1; at the same time we should expect that T_1 does not go beyond $\mathcal{M}$ in any essential respects. It is worth elaborating what is required, following the criteria for formalization set forth in [Feferman 1979] pp. 171–72, for any $\mathcal{M}$ and T, as follows.

(i) T is an *adequate formalization* of $\mathcal{M}$ if every concept, argument and result of $\mathcal{M}$ may be represented by a (basic or defined) concept, proof and theorem, resp. of T.

(ii) T is in *accordance with (or faithful to)* $\mathcal{M}$ if every basic concept of T corresponds to a basic concept of $\mathcal{M}$ and every axiom and rule of T corresponds to, or is implicit in, the assumption and reasoning followed in $\mathcal{M}$ (in other words, if T does not go beyond M conceptually or in principle).

The idea of T being *directly adequate to*, resp. *directly in accordance* with $\mathcal{M}$ is clear. We would say that T is *indirectly adequate to* $\mathcal{M}$ if a theory directly adequate to $\mathcal{M}$ is reducible to T in an elementary way (e.g. by a translation or proof-theoretic reduction) while it is *indirectly in accordance with* $\mathcal{M}$ if T is reducible to a theory directly in accordance with $\mathcal{M}$. There is a second way in which a theory T may be indirectly adequate to $\mathcal{M}$: that is to reformulate the concepts, proofs and theorems of $\mathcal{M}$ informally in such a way that the resulting $\mathcal{M}'$ can be directly formalized in T.

Obviously these criteria are not precise and there may be reasonable differences of opinion as to their application in specific cases. The idea, again, is to say what strikes us as a just ascription on the basis of general experience. Detailed work of formalization may then lead us to modify such an attribution. In particular, it is a common result of such work that a system T which appears to us to provide an adequate and faithful formalization of a body of mathematics $\mathcal{M}$ goes far beyond what is actually needed to represent $\mathcal{M}$ in practice.

Consider, first, elementary (non-analytic, non-algebraic) number theory $\mathcal{M}$. At first sight, this has PA as an adequate and faithful formalization in the way we have presented it in the language L_0 with symbols for all primitive recursive functions and (thence) relations. But evidently in practice one makes use of only a small stock of these, e.g. $+$, $\cdot$, $\exp, \Sigma, \Pi, p_i, <$, $|, \equiv$, etc. The fact from [Gödel 1931] that we can make do with only $+$ and $\cdot$, shows the indirect adequacy of *that* restricted version of PA to informal elementary number theory. While the general principle of induction is in accordance with this body of mathematics, detailed work of formalization shows that the complexity of inductive arguments used in

[15] With apologies to Hao Wang for borrowing the title of [Wang 1974], but with a different meaning.

practice is very low and rarely goes beyond Σ^0_2-IA (or, equivalently, Π^0_2-IA), i.e. low parts of PA are still directly adequate to $\mathcal{M}$. Then for more specific results or groups of results one can verify that the system Σ^0_1-IA is adequate to a significant portion of $\mathcal{M}$ and hence, by Theorem 1, that PRA is indirectly adequate to that part of elementary number theory.[16] For recent case studies and extensive references to earlier work, cf. [Hájek and Pudlák 1993].

Turning next to analysis, we find a much more complicated picture, both as to the categorization of mathematical practice and choice of related formal languages and systems. One immediate way to break up practice so as to begin to make this more manageable, is by means of the separation between concrete analysis, based on finite-dimensional real and complex spaces, and abstract analysis, based on various kinds of general spaces such as metric, Hilbert, Banach, etc.

In considering the direct formalization of concrete analysis, one must be able to deal among other things with:

(i) the basic number systems (integers, rationals, reals, complexes),

(ii) finite and infinite sequences of numbers,

(iii) specific operations such as sum and product of finite and infinite sequences,

(iv) sets and functions of numbers (n-ary, for various n),

(v) specific operations on functions and infinite sequences of functions such as differentiation and integration, and

(vi) specific operations on sets and infinite sequences of sets, such as complement, union and intersection.

Taking first-order (type 0) arithmetic as basic, and considering real numbers to be represented as sets or sequences of rationals and thence (by reduction to N) as sets or sequences of natural numbers, we need to consider reals as second-order (type 1) objects. Then functions and sets of reals are located at the next type level (third-order or type 2 objects), and specific operations on such are located at still one type higher. Thus for the direct representation of concrete analysis, the language of functions and sets of finite type is adequate, but already the part with first-, second- and third-order variables serves most purposes comfortably. The question then is, what principles (axioms, rules) are appropriate for this formalization?

On the face of it, concrete analysis makes use of the ideas of "arbitrary" set and "arbitrary" function (of n-tuples of numbers) and, for these, the general forms of the Comprehension Axiom CA and the Axiom of Choice AC, would seem to be the appropriate set and function existence principles. But what special forms of these principles are actually required for

[16]Dedicated finitists, beginning with [Skolem 1923], had made an effort to see what part of elementary number theory could be directly formalized in PRA; cf. also [Goodstein 1957]. However, it is much easier to work directly in Σ^0_1-IA to see what can be justified by PRA.

and/or in accord with practice is only revealed by closer attention to the details of formalization. In fact, concrete analysis deals with relatively narrow classes of functions and sets, such as piece-wise continuous functions (or, more generally, Lebesgue measurable functions) and open and closed sets (or, more generally, Borel sets of finite level), whose properties are individually determined by a countable amount of information. Such functions and sets may then be represented or "coded" by second-order objects, and thus the second-order system CA (or AC) is indirectly adequate to concrete analysis. The recognition of this is often credited to [Hilbert and Bernays 1939], Supplement IV. However, it was already essentially realized by Weyl in his monograph [1918] that far weaker assumptions about second-order objects, given by arithmetical closure conditions, i.e. on the order of ACA, account at least for the familiar (19th c.) concrete analysis of piece-wise continuous functions.[17]

Modern continuation of Weyl's work has given much more precise information about what can be done in ACA and related systems. Here we take note of the result that as formulated with the second-order axiom of induction I_1, ACA$\restriction$ is reducible to PA. Now [Feferman 1977] and [Takeuti 1978] described theories of finite type extending ACA$\restriction$ which are still reducible to PA, and in which the concrete analysis of piece-wise continuous functions and of more general classes of functions (e.g. Lebesgue measurable) can be carried out directly. These systems have not been described explicitly in §2 above, so we can't plug them in as examples for the scheme $(*)$; for that, cf. [Feferman 1988]. On the other hand, considerable detailed work on the formalization of analysis in second-order systems has been carried out by Friedman and Simpson and their students within Friedman's "Reverse Mathematics" program; cf. [Simpson 1987] for a survey.[18] This achieves its end by reformulating and/or coding various higher-type notions in second-order terms and is thus an example of the second way (described above) in which a theory T may be indirectly adequate to a body of mathematics.

The work of the Friedman-Simpson school gives quite precise information about what principles are needed to prove various statements of concrete (and even abstract) analysis, when these can be reformulated in second-order terms. Combined with the earlier work mentioned above, the general conclusions are as follows.

(i) That part of concrete analysis which does not require set-theoretical (transfinite) notions and constructions in any essential way can be comfortably formalized in systems like ACA$\restriction$ which are reducible to PA.

(ii) Those parts of analysis that require essential use of transfinite ordinals such as "descriptive" set theory of the reals (Borel and projective hierarchies, etc.) can be mostly carried out in systems reducible to Π^1_1-CA and at worst in Δ^1_2-CA or Δ^1_2-CA + BI.

[17]Cf. [Feferman 1988a] for an exegesis of the principles which underly Weyl's work.

[18]Full details will become available in a book on subsystems of second-order arithmetic in preparation by Simpson.

On the other hand, a striking result of the work on the Reverse Mathematics program is that:

(iii) A great deal of what can be done in ACA$\restriction$ can already be formalized in the system RCA+WKL+Σ^0_1-IA, which is reducible to PRA by Theorem 3 of §2.6.

Abstract analysis goes beyond concrete analysis by its use of arbitrary spaces of various kinds, e.g. metric spaces, Banach spaces, Hilbert spaces, etc. Moreover, for functional analysis, applications often involve spaces of functions such as the L^p spaces, and, more abstractly, spaces of functions between give spaces. Thus the mathematics *prima-facie* requires *variable types* $X, Y, \ldots$ to represent talk about arbitrary spaces (of the kinds indicated) and higher-type constructions such as of the type $X \to Y$ of all functions from X to Y, from which function spaces are constructed. It happens that for most of the applications of abstract analysis one deals with separable spaces, i.e. those X for which there is a countable dense subset X_0 whose completion (under a suitable metric or norm) is X. Talk about such X may informally be recast by talk about X_0, and the latter, being countable, may be coded as a subset of the natural numbers. It is in this way that parts of abstract analysis are indirectly accounted for in the Friedman-Simpson approach. However, a direct formalization requires something like the theories of variable finite type VT$_\mu\restriction$ or W described in §2.10, again reducible to PA by [Feferman 1985]. More information about the part of concrete and abstract analysis that is directly accounted for in this way is given in that paper and in [Feferman 1988a]. There I conjectured that *all of scientifically applicable mathematics can be directly formalized in* W; further discussion of this conjecture further discussion will be found in the paper [Feferman 1993].

What, finally, is to be said about the philosophical significance of this work? It seems to me that the information provided by the kind of case studies described here involving the formalization of considerable tracts of everyday mathematics in appropriate systems, in combination with the results of their proof-theoretical reductions such as those presented in §2, must be taken into account in the continuing efforts to develop a relevant and sustainable philosophy of mathematics. To take just one example, the Quine-Putnam (scientific) indispensability arguments for a form of mathematical realism ([cf. Maddy 1992]) are considerably weakened when confronted with the kind of information described in the preceding paragraphs.[19] The question in consequence of that is whether the indispensability arguments retain sufficient force to be maintained in the arsenal that has been mounted in defense of realism in mathematics, as, for example, by [Maddy 1990].

In general, the kinds of results presented here serve to sharpen what is to be said in favor of, or in opposition to, the various philosophies of mathematics such as finitism, predicativism, constructivism and set-theoretical realism. Whether or not one takes one or another of these philosophies seriously for ontological and/or epistemological reasons, it is important to know which parts of mathematics are in the end justifiable on the basis of the respective philosophies and which are not. The uninformed common view – that adopting one of the non-platonistic positions means pretty much giving up mathematics as we know it – needs

[19] I argue this at greater length in [Feferman 1993].

to be drastically corrected, and that should also no longer serve as the last-ditch stand of set-theoretical realism. On the other hand, would-be non-platonists must recognize the now clearly marked sacrifices required by such a commitment and should have well thought-out reasons for making them. Though I personally believe that the kind of results described here on the whole strengthen the case for a non-platonistic philosophy of mathematics and further undermine the case for set-theoretical realism, they do not speak for themselves to that extent, and it is at that point that well-informed critical philosophical discussion must take over.

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