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### The Creation of Numbers from Clay: Understanding Damerow's Theory of Material Abstraction

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**Abstract:** We are responding to an interpretation of the origin of numbers in Mesopotamia by K. A. Overmann. She has challenged the idea that the emergence of numbers can be understood in terms of abstraction and, in this context, has criticized the work of Peter Damerow on this topic arguing that it is psychologically implausible and outdated. We clarify Damerow's approach by explicating the concept of abstraction and demonstrate its relevance for current discussion by formulating a thesis on the origin of numbers within the framework of Damerow's theory.

**Keywords:** abstraction, representation, origin and development of numbers, Cuneiform, historical epistemology, cognitive archaeology, material engagement

#### §1. Introduction

§1.1. Peter Damerow is known to readers of this journal as the co-founder, with Robert K. Englund, of the Cuneiform Digital Library Initia*tive* (CDLI). In the 1980s, as a pioneer of the Digital Humanities, he began to develop computerbased methods of analysis to decipher the counting systems of early Mesopotamian mathematics in comparative procedures. He complemented this empirical work on the archaeological material with conceptual work on models of the historical genesis of the concept of number and mathematical thinking in general (a representative selection of his papers in English translation can be found in Damerow 1996a, for an appraisal of his work see Renn and Schemmel 2019). In recent years, these works have been subjected to severe criticism. Karenleigh Overmann, who works on questions of early number development and to whom we owe some of the most important recent work on this subject (cf. Overmann 2021b), has formulated fundamental objections to Damerow's conceptual model, published in par-

ticular in the pages of this journal (Overmann 2018). To today's readers, therefore, it could seem that Damerow, while having great historical merits, is now largely obsolete and outdated when it comes to the current theoretical discussion. In this article, we shall directly contradict and correct this notion and show that Damerow's ideas are relevant to current research at the intersection of historical epistemology and cognitive archaeology and that they can lead to fruitful hypotheses and insights in a field of research that is highly topical, but in which people from many different disciplines are still debating about the basic concepts (cf. e.g. Núñez 2017; d'Errico et al. 2018; Nieder 2019; Schlaudt 2020; Overmann 2021b, and many more). To this end, in this article we (1) examine the background of Overmann's critique of her predecessors Schmandt-Besserat and Damerow, (2) briefly contextualize Damerow's approach in order (3) to reconstruct his model of abstraction. On this basis we shall make a conjecture about the genesis of "abstract counting" with the twofold aim of contributing to our understanding of the nature and origin of numbers and of practically demonstrating the relevance and fruitfulness of Damerow's approach in the contemporary context.

## §2. The tale of abstraction from concrete counting

#### §2.1. Schmandt-Besserat's account of the emergence of "abstract numbers" from "concrete counting"

**§2.1.1.** Denise Schmandt-Besserat, an archaeologist, whose work on the origins of writing in Mesopotamia has been at the center of discussion for decades, has summarized her position on the origin and development of numbers in a collection on the history of measurement by comparing the use of tokens to count goods in the administration of Sumerian temples with what she called "concrete counting" as still practiced in various parts of the world:

Concrete counting is characterized by different sets of number words - numerations - to count different items. Sets of words of our own vocabulary, such as 'twin, triplet, quadruplet' or 'solo, duo, trio, quartet' referring to children of a common birth and groupings of musicians, may help explain the concept of concrete counting. Namely, a word like 'solo' fuses together two concepts, 'one' and 'musician', without any possibility of separating them. The same was true for tokens. For example, one ovoid token [...] stood for 'one jar of oil' without the possibility of splitting up the notion of number 'one' with the notion of the object counted, 'jar of oil'. Because these two types of information could not be abstracted from each other, numerosity was expressed in one-to-one correspondence. Three jars of oil were represented by three ovoid tokens – literally 'one jar of oil', 'one jar of oil', 'one jar of oil'. The token system illustrates therefore a technique of counting fundamentally different from ours. There were no tokens to express 'one', 'two', and 'three', independently of what was being counted. But instead, as is typical of concrete counting, each token type counted exclusively a specific category of items: ovoids could count only jars of oil and jars of oil could only be counted with ovoids. (Schmandt-Besserat 2010, 28–9)

**§2.1.2.** She continues:

The fact that pictographs, such as that standing for 'jar of oil', were never repeated in one-to-one correspondence signals a radical change in counting. The sign for 'jar of oil' was preceded by numerals – signs for 1, 10 and 60 . [...] Pictography thus marks the extraordinary event when the concept of number was abstracted from that of the item counted. (Schmandt-Besserat 2010, 31)

§2.1.3. Schmandt-Besserat's idea that the use of tokens led to proto-cuneiform gave "a revolutionary new twist to the narrative on how writing was born", Valério and Ferrara (Valério and Ferrara 2022, 36) explain, because "counting, in this view, and not pictures, led to the emergence of writing." But even though one accepts this framework, the details of the story as described in the above quotes are not beyond doubt, for if taken overly literally such a position would assert that we could not understand the reference to a "quintet" of basketball players as a reference to a team. And it would expect us to believe that a Sumerian shepherd, who forgot his bag of sheeptokens, would have been incapable of using the goat-tokens he happened to have in his pocket to count the sheep this one time. Such excursions from archaeology into speculative historical cognition research have called out criticism - some of it also overshooting the mark by including more sanguine positions in the critique.

#### §2.2. Overmann's critique

§2.2.1. In a recent paper reacting to Schmandt-Besserat, Karenleigh Overmann sees the need for a new terminology to deal with numbers, especially in Near Eastern studies, "where for decades, the labels 'abstract' and 'concrete' have dominated how the archaic number systems of Mesopotamia have been understood, chiefly through the work of archaeologist Denise Schmandt-Besserat and psychologist Peter Damerow." (Overmann 2021a, 292) "Numbers, she insists, "do not start out 'concrete' to become 'abstract' at some later point in time." Numbers are said to be abstract as concepts, but also concrete insofar as they depend on material forms: "This dependence on material form remains true even for highly elaborated numbers mediated by notations." With regard to Damerow in particular she continues: "Damerow saw token-based accounting as involving concrete numbers and the emergence of writing as enabling the development of abstract or 'secondorder' numbers, thereby differentiating 'material means' from 'conceptual structures' but ignoring the materiality of writing" (Overmann 2021a, 292, 293). Overmann's introduces her own counterprogram:

[I]t seems time to retire the labels 'abstract' and 'concrete', for they inaccurately characterize what numbers are as concepts, incompletely describe the process, mechanisms, and states of conceptual change in numbers, and evoke outdated concepts of progressive cultural evolution (Overmann 2021a, 294).

[...] rejecting the idea that tokens were confused with the objects enumerated or that they represented concrete numbers while written notations meant abstract ones. Further, the mechanism of change in social thinking may not have been ontogenetic maturity [as assumed by Piaget and, according to Overmann, adopted by Damerow – the authors] but, as argued here, material engagement: Numerical content and structure were influenced by the material forms used for counting, associated behaviors, and psychological processing related to the acquisition of an elaborated cultural system (Overmann 2018, 5).

[...] thinking of number concepts in terms of abstractness and concreteness misunderstands their nature, which is always a mix of neural activity, material forms instantiating numerical information, and behaviors interfacing the two, independent of the degree of elaboration achieved in any particular number system. What the comparison shows is how the incorporation of different material forms alters the respective contribution of these components (Overmann 2021a, 315).

**§2.2.2.** The main problem in the analysis of "concrete counting," that is of counting with counting systems that are different according to the kind of objects counted, seems to be that Schmandt-Besserat's account focuses on written numerals as the only material vestiges of the use of numbers in ancient times. But, Overmann suggests, if we take evidence from ethnology as a proxy for neolithic cultures, the picture changes, because then we encounter a mismatch between quite well developed oral counting practices (going up to numbers of

25 or beyond) and a theory based on the writing practices according to which the same people had difficulties to abstract numbers as small as one, two, and three from concrete bundles of objects.

**§2.2.3.** As Chrisomalis (Chrisomalis 2005, 4, quoted in Overmann 2021a, 301n78) puts it:

If [...] polyvalence and contextdependence imply an absence of abstract number concepts, then paradoxically, the quasi-literate Uruk accountants would be less numerate than the average Sumerian who did not use texts, only number words.

**§2.2.4.** And Chrisomalis (Chrisomalis 2010, 502) objects:

the accountants and scribes who used [tokens and numbers] were able to manage complex administrative tasks, and it is implausible that they did not recognize that '8 sheep' and '8 bushels of grain' had something in common.

**§2.2.5.** For Polynesian counting, which applies category-sensitive counting systems, Overmann comes to the conclusion that "[r]ather than being dissociated or separately developed, the different counting sequences were derived from a single method of counting" (Overmann 2021a, 301); and she suggests that something similar holds for Proto-cuneiform counting (Overmann 2021a, 310, fig. 6) Valério and Ferrara (Valério and Ferrara 2022, 39) confirm that "no compelling example has been provided of a language whose abstract number words etymologically derive from concrete number words, which would support the proposed unidirectional development."

#### §2.3. Motivation and outline of our reply

**§2.3.1.** While there are certainly reasons to sympathize with Overmann's critique of Schmandt-Besserat's sometimes exaggerated conclusions, we think that her inclusion of Damerow's quite different position in this critique is mistaken and serves not to clarify the issues but rather to confuse them. Importantly, a number of crucial conceptual distinctions made by Damerow are ignored, and basic concepts are conflated – and this has consequences for the conclusions drawn. The point here is not to defend the person, Peter Damerow, but to clarify the approach to the concept of number that he advocated – a concept that had a number of presuppositions common in

the fields in which he was working. Damerow, who was a mathematician by training, had been engaged in research on the psychology of learning mathematics and the didactics of teaching it for twenty years before he entered into interdisciplinary projects on the history (including the earliest history) of mathematics. During this time, he was also an avid reader of ethnological literature on mathematical knowledge in non-western cultures. Learning mathematics and dealing adequately with quantities was something common to European children, Polynesian and Indonesian villagers and ancient Sumerian scribes: all learn techniques of counting and can engage in reflective abstraction about these practices. The role of the material means used to count and to deal with quantities was always at the focus of Damerow's attention. And this interest included especially the social practices in which these means were used and passed on to future users. Thus, in many regards Overmann is advocating directions of research already started on by Damerow.

**§2.3.2.** Let us begin with the correction of a few misunderstandings:

- Damerow did not assert that "tokens were confused with the objects enumerated" or that "tokens represented concrete numbers"
- he did not think that ontogenetic maturity is the mechanism of change in social thinking
- he did not ignore the materiality of writing, nor did he deny that even elaborate numbers depend on material forms
- he never speaks of "concrete numbers" or "concrete counting,"<sup>1</sup> and while the term "abstract number" does occasionally occur, this is because numbers are *per se* abstract.
- he also does not speak of "second-order numbers" but rather of "second order representations" and of "number" as a "second-order concept".

**§2.3.4.** If we want to understand the development of the abstract concept, number, out of techniques of counting and keeping track of concrete objects, we need a more precise analysis of the role of the materiality of representations and the process of abstraction metaphorically called "internalization." In our opinion, Damerow's analysis is still the most promising starting point for a better un-

derstanding of the origin of numbers, including the project Overmann envisions. It is important to distinguish Damerow's systematic efforts in the cognitive science of mathematics learning, from unfounded speculations – even when these are coupled with innovative archaeological discoveries. Therefore, it seems worthwhile to present certain aspects of Damerow's approach and to examine its potential.

## §3. Background: From Historical Epistemology to Cognitive Archaeology

#### §3.1. Historical Epistemology: Studying material means for understanding cognitive content

§3.1.1. Damerow was engaged in a project often called historical epistemology, the historical study of the cultural forms and determinants of knowledge acquisition and tradition. While the term has numerous and divergent interpretations, Damerow's version of the project can be characterized by a focus on the material means of representation and the social forms of practice in which they are embedded. "Social forms of practice" are a direct legacy of the concept of labor in the approach of Historical Materialism by Karl Marx and Friedrich Engels, where Damerow also had intellectual roots. Damerow stated programmatically: "According to historical materialism, the socio-historical development begins with labor, and it is labor that moves this development forward" (Damerow 1996a, 383). Applied to the question of hominization, this means that "anthropogenesis is the result of self-creation by means of concrete labor" (Damerow 1996a, 391). Concrete labor in general is labor with tools ("the tool, as objectified labor," Damerow 1996a, 393-4), which includes both the common material tools and machines, but also pictograms, diagrams, signs and writing, i.e., the "tools of mental labor" (Damerow 1996a, 398). Hominization, prehistory and history, in turn, also comprise the dimension of the development of cognition, its forms and contents, which raises the question of "how cognition developed in the context of labor" (Damerow 1996a, 395), that is, in particular how cognition evolved and still evolves in relation to the material and symbolic tools. For the more specific project of the history of science, this raises the task of understanding how new knowledge is created through the systematic exploitation of the

<sup>&</sup>lt;sup>1</sup> Take the book, Damerow 1996a: "number" has 1011 occurrences, "concrete number" 0, "concrete counting" 0, "abstract number" 26.

possibilities inherent in the material and symbolic tools available for experimentation, representation and calculation: "When using a tool, more can always have been learned than the knowledge invested in its invention" (Damerow 1996a, 401). This can be read as an implicit definition of historical epistemology as it was understood by Damerow and as it also informed the original research program of much of the Max Planck Institute for the History of Science in Berlin.

§3.1.2. In his efforts to find answers to these questions, Damerow made frequent references to the Swiss developmental psychologist Jean Piaget, which today provoke criticism from Overmann in particular, since Piaget is largely considered outdated as a psychologist. This is not the place for a fundamental analysis of the difference between reading Piaget as an empirical developmental psychologist and reading him as a genetic epistemologist. For Damerow, Piaget's genetic epistemology served as a model for a naturalistic reconstruction and explanation of forms of knowledge. Cognitive structures, as Damerow's Piaget conceived them, are internalized coordinating schemata of actions. It is not Piaget's biological - perhaps even tendentially Lamarckian - form of developmental theory that is of interest, but rather the establishment and characterization of various stages of development that can also be reconstructed by a cultural-historical analysis, in which reflection on practice with the particular means of representation leads to cognitive change. Damerow's goal was to develop the historical epistemological categories needed to understand the development and tradition of mathematical skills as well as the cognitive processes that enable us to develop or learn abstract concepts. In this context, mathematical or quantitative concepts are not taken to be abstracted directly from the objects in the world, but from the coordination of actions that deal with such objects. With Piaget, the emergence of mathematical mental operations is seen as due to the internalization of systems of real actions. The important point is that the step of abstraction that occurs in a cognitive practice with material objects or with material representations of objects occurs in the practice itself and then is internalized - whereby *internalization* in the end is just a placeholder for the ability to perform a cognitive practice normally done using material representations without the actual presence of those means. The paradigm of internalization is the performance of mathematical calculations in one's head. When

we assume that an experienced Sumerian shepherd, who lost his string of tokens, could probably deal somehow with 8 sheep and 11 goats as 19 animals by performing some operations "in his head," the notion of internalization postulated is basically metaphorical and functionalistic; but it does not ignore the materiality of the tokens (or fingers) that made the calculations possible in the first place. Contrary to Piaget, Damerow does not take these basic structures to be epigenetically determined, but rather developed in interaction with culturally specific challenges and conditions of action (Damerow 1996a, 255–6).

#### §3.2. Historical Epistemology, Cognitive Archaeology, and Material Engagement

**§3.2.1.** As just explained, Damerow developed his account of the emergence of mathematics in the context of Historical Epistemology. Our current discussion, however, is situated in the conceptual framework of Cognitive Archaeology and, in particular, the theory of Material Engagement. For our clarification of Damerow's position it might be helpful to understand how the two conceptual frameworks relate to each other.

**§3.2.2.** Cognitive Archaeology is a subdiscipline of Archaeology that aims at reconstructing, on the basis of the remains of the material culture, the cognitive space and cultural capacities of prehistoric groups (Haidle 2015, 863, de Beaune 2011; Renfrew 1993). This includes, in a narrower sense, the cognitive skills directly involved in the production and use of tools, as well as, in a broader sense, the products of intellectual activity such as cosmology, religion and ideologies. The theory of material engagement (Malafouris 2013) builds a bridge between material culture and cognition with a view to providing an account of how material culture may have contributed to the emergence of cognition. The possible roles played by material culture range from its being a mere catalyst for biological and cultural evolution to its constituting an active component of cognition in the sense of extended cognition.

**§3.2.3.** There is an obvious difference of scale between cognitive archaeology and the history of science. Whereas the latter is typically concerned with a scale of decades and centuries, the former deals with a development covering hundreds of thousands of years; whereas history of science investigates changes in cognitive content, cognitive archaeology also studies the development of basic cognitive skills and thus may also deal with cases of gene-culture coevolution, where cultural developments alter the biological selection pressure and thus leave a footprint in the human genome, and with cases of the co-option or exaptation of existing neuronal resources.

**§3.2.4.** In the case of the emergence of mathematics in neolithic Mesopotamia these differences become less important. But not only do the history of science and cognitive archaeology merge seamlessly here, since their respective time scales begin to converge in this case; we can furthermore note that historical epistemology and the theory of material engagement, as specific approaches in both fields, start from very similar, if not identical, methodological assumptions, which can be summarized in the thesis of the constitutive importance of material artifacts for human cognition, its form and its content. When it comes to the special case of semiotics, which is the one we are dealing with (and for which Historical Epistemology is well equipped, since one of its roots is Derrida's semiotics), the thesis of Material Engagement opposes itself to the "representational" paradigm which tends to reduce symbols to mere expressions of preexisting meanings, originating from the human mind. Against this approach, Material Engagement insists on the importance of material agency; it wants to "take into account how the physical properties of the medium of representation affect the semiotic process"; and it defends the thesis that "[m]eaning is the temporally emergent property of material engagement, the ongoing blending between the mental and the physical" (Malafouris 2013, 117). For the case of the emergence of numbers from neolithic clay artefacts, Malafouris defines the following program which we quote in extenso because we can use it as a sort of benchmark for our reconstruction of Damerow's theses:

The agency of clay, in all its different manifestations, is not to be found in the way it represents number but instead in the way it *brings forth* the concept of number. The clay token as enactive sign is a constitutive part of what it expresses, which otherwise cannot be known [and] could not have been achieved by the naked brain. In other words, the tangible material reality of the clay token as an 'epistemic' artifact enabled the already evolved parietal system to support approximate numerosity, by getting reorganized, and thus partially 'recycled,' to support also the representation of number. [...] No doubt the representational properties of neural networks, like those that subserve numerical thinking, become realized inside the head, but in this case the systemic properties of the cognitive structures from which they derive extend beyond skin and skull. (Malafouris 2013, 116)

## §4. Damerow's account of abstraction: abstraction through representation

# §4.1. What does "abstract number" mean, and what needs to be explained in a theory of abstraction?

**§4.1.1.** Before looking at Damerow's account of abstraction, i.e. his account of *how* abstraction works, let us try to understand *what* the word "abstract" means according to Damerow and *where* the process of abstraction leads us.

§4.1.2. As remarked earlier, Damerow occasionally, but very rarely uses the term "abstract number(s)." The reason probably is that for Damerow numbers simply are abstract entities, and in most contexts this need not to be explicitly stated. That numbers are of abstract nature, however does not mean that they are mental (or even metaphysical) objects. There is no doubt that Damerow thought of numbers as objects that are "dependent on material forms" (as Overmann puts it), and that he did this in a very strong way. The historian of accounting, Richard Mattessich, once expressed a similar idea in words that seem quite helpful, since they draw our attention to how material tools are embedded in social forms of actions and unfold their potential meanings there:

The term 'abstract token' might be confusing because those complex tokens are still concrete clay objects, but *now they are used in a way that approaches numerals in the abstract sense.* Thus the term 'abstract' does not refer to the token itself but to its use. (Mattessich 1987, 79)

**§4.1.3.** Damerow's conception of numbers becomes clearer in the following quotation, which is one of the rare occasions where he spoke of "abstract numbers," thus emphasizing their abstractness:

Imagine the situation in a cultural setting without any objects that could usefully be iterated, a community in which everybody knows his pigs, his neighbors, and his tools individually. In this situation how could one present a problem that on the one hand is meaningful and on the other has as its solution an abstract number? Why should somebody count if there is nothing to be counted? It makes much more sense to assume that there have to be iterated objects in a culture before any arithmetical activity can emerge that iterates symbols. (Damerow 1996b, 142)

§4.1.4. If we take this as an implicit definition, abstract numbers are numbers that are used even if "there is nothing to be counted." While this definition may well be quite consistent with a common notion of abstractness, it nevertheless changes in a significant way the problem that an abstraction theory of numbers confronts as applied to the archaeological record from Mesopotamia. Seen through the lens of this definition, the task does not consist in abstracting, say, the number 3 out of the expressions "trio" and "triplet," to use Schmandt-Besserat's example, or to recognize that '8 sheep' and '8 bushels of grain' "have something in common," as Chrisomalis said facetiously. The task rather is to explain how someone who knows how to count and who even might be in possession of a general counting scheme from which he or she derives object-specific counting techniques, as Overmann stipulates quite plausibly both for Polynesia and Mesopotamia - how such a person can come to use numbers without counting anything in particular and how he or she can still assume that he or she is doing something that makes sense. That this criterion is not trivial can be directly seen by comparing number words to other instruments of human action. One cannot pound with a hammer without hitting something particular (and even if one uses the hammer for other purposes, for instance as a lever, one still stems some determinate object with it). So why should it make sense to count without counting something? Why are number words different?

**§4.1.5.** It is quite obvious – and will become more so in what follows – that this notion of abstractness is a non-mentalistic one. Dealing with abstract numbers might induce reflection and the emergence of a new meta-vocabulary such as the word "number" itself (and it is in this sense that Damerow spoke of numbers "as second-order concepts," cf. Damerow 1996b). But abstraction is not in itself a mental activity. Having said this, we are not entirely sure how different from Overmann's own notion this notion of numbers

as abstract objects is. At one point she mentions the properties of "entitivity" and "relatedness" as constitutive qualities of abstract numbers:

As thus narrowly defined, the distinction [between 'concrete' and 'abstract' numbers] hinges on two things: whether or not numbers are conceived as objects or entities in their own light (the property of *entitivity*), rather than as properties of quantity or collections of objects, and whether numbers relate to each other or to the objects they count (the property of *relatedness*). (Overmann 2021a, 296–97)

§4.1.6. At the same time, however, it is not completely clear to us whether she shares this account. She explicitly "challenges the idea that 'number' is a monolithic construct, one identical to what contemporary Western numbers are today" (Overmann 2021a, 315), and it seems that the monolithic view is an implication of the foregoing notion of abstract numbers. On the other hand, she seems to adhere to this notion of "abstract" numbers when she writes: "Separating numbers from the items they counted by means of material proxies [...] helped influence numbers toward being conceived as objects more defined by their relations to one another than the things they enumerated, influencing numbers toward becoming a relational system." (Overmann 2021a, 313) And: "As this phenomenon transformed early writing into a system of literacy, it also helped influence numbers toward being conceived as entities (as represented and influenced by single signs) rather than collections of objects (sets of tokens), even though both are composed of multiple elements" (Overmann 2021a, 313).

**§4.1.7.** We also have doubts as to whether an understanding of numbers as independent objects or entities is really a necessary component of the phenomenon of "abstract" counting that interests us (in the sense of "counting without counting anything in particular"), or whether it does not actually belong to a later stage of abstraction. Here we will only deal with the first step for the moment. In subsequent paragraphs we will attempt to show how exactly the process of abstraction operates according to Damerow and how we propose to apply this model to the problem of counting and numbers. In the course of these discussions, it will also become clearer what exactly is meant by "abstract" in general and "abstract numbers" in particular, why abstraction is not a mental process and why materiality is constitutive for abstraction and thus also for mathematical thinking in particular.

## §4.2. The fine mechanics of abstraction according to Damerow

**§4.2.1.** Damerow's most detailed (though not always perfectly clear and easy to understand) account of abstraction can be found in a paper written in German, "Preliminary Remarks on a Historical Epistemology of the Development of the Notion of Number" (Damerow 1994) and which unfortunately was not included in the collection of his essays in English translation (Damerow 1996a). In what follows, we will primarily use this article, which, also seems to us to be fully consistent with what is available to a non-German-speaking readership through the collection of essays. All quotations from this paper are our own translations.

**§4.2.2.** Damerow thinks of abstraction as rooted in human action, more precisely in the socially organized exchange with nature through labor. However, the things of everyday life and the objects of work as such cannot be the starting point for abstractions, because "they do not gain any meaning beyond their physical perception" ("sie gewinnen keine über ihre physische Wahrnehmung hinausgehende Bedeutung", Damerow 1994, 272). Perhaps this idea can, following Sigaut (Sigaut 2012), be spelled out in Gibsonian terms: it is precisely due to their affordance that the objects of everyday life become "invisible" and fuse seamlessly with the environment, but do not become the starting point for abstractions.

§4.2.3. In what context can abstractions, such as are presumably latent in all social forms of practice, unfold and modify the dynamics of cultural and cognitive development? Here is Damerow's proposal: "A powerful means of abstraction, I should like to claim, is *representation*" (Damerow 1996a, 373). This idea came about, of course, in Damerow's study of the archaeological evidence. The tokens that are enclosed in clay bullae which were used in Mesopotamia of the midfourth millennium BCE are a case in point. Note that Damerow does not claim that the originators of this representational technique were unable to count or didn't possess a general method of counting from which object-specific counting systems were derived (in the sense of Overmann 2021a). But when it comes to recording economic transactions, the pragmatic requirements are obviously

quite different, because the issue now is to *store* information. For this reason, oral counting (which may be highly developed) is excluded, at least as long as a technique of writing down spoken language (i.e. a phonetic alphabet) is lacking; instead, the technique of one-to-one assignment of tokens, which, from a purely cognitive point of view, is more primitive, is employed, parallel to and not entirely reducible to oral counting.

§4.2.4. For Damerow, such representations of concrete objects, which he calls "first-order representations," are the starting point of abstraction, or, more precisely, not the material representations themselves, but the "coordination of actions applied to them and transformed by them in some way" (Damerow 1994, 255). With the material representatives, "essentially the same actions can be performed as with the real objects themselves" (Damerow 1994, 261), but these actions can be "performed more easily" (Damerow 1994, 263). This seems obvious: the same spatial rearrangements can be performed with a set of tokens as with a flock of sheep or a class of school children represented by them. But unlike sheep, tokens do not need to be herded, nor do they get sick or pregnant. And unlike the situation in a school class, one can put arbitrary individuals next to each other without the situation getting out of control. Retrospectively, we can say that by means of their representations, the groups are reduced to mere bearers of their cardinality, while the real objects - sheep or pupils - are rather judged according to the parameter of "manageability," in which cardinality does not enter in a definite way: with increasing size control tends to become more difficult, but in a group of fixed size the controllability may well depend on the internal arrangement and on the surrounding environment, as teachers and shepherds know. Elimination, or at least considerable reduction, of internal dynamics and external context are the two crucial aspects of this kind of representation. Damerow also emphasizes that representations are "more general" than the objects represented. Tokens which happen to represent sheep could also be used to represent cows (Damerow 1994, 272, Damerow 1996a, 378). Altogether, tokens used as representations of concrete objects embody a "potential abstraction" (Damerow 1994, 272). Here abstractness does not denote anything mental, but the characteristics of a particular material arrangement: reduced internal dynamics, reduced context, general applicability.

**§4.2.5.** For this potential abstraction to become real, a further step is needed. The story Damerow tells us is indebted to Piaget:

The emergence of the mental operations of logical-mathematical thinking is based [...] on an internalization of systems of real actions [with the tokens or, in general, first-order representations], through which they become elements of reversible mental transformations. [...] The meta-cognitive constructs generated by reflexive abstractions, that is, the logical-mathematical concepts, to which the number concept belongs in particular, can thus be conceived as internally represented invariants of transformations to which the objects are subjected when we handle them [im handelnden *Umgang*]. (Damerow 1996b, 256, cf. also Damerow 1996a, 304)

**§4.2.6.** Cardinality is indeed an invariant relative to all possible spatial rearrangements of objects or, easier, their material representatives (tokens).

§4.2.7. As we can see from this quotation, the process of internalization is a crucial step. However, it is of utmost importance to understand that this step is neither synonymous with abstraction nor should it be understood as a bridge leading from the realm of the material to that of the mental. Let us briefly make both points clear. First, internalization does not mean abstraction, because the abstraction has already been completed at this point of the story. We are concerned with a material abstraction that occurred *through* representation and which can at best be described as being "reproduced" in internalization, but not brought forth. Secondly, it is misleading to understand internalization as building a bridge towards the mental. Internalization merely means that a cognitive performance that was previously tied to a specific external, material medium can now be performed, at least in part, also in the absence of this medium. What changes is the relative weighting of the different cognitive abilities involved, particularly perception, manual operations and working memory. In philosophical terms, however, we are dealing with processes of the same kind, not with a material process on the one hand and a mental process on the other. Exactly which processes are behind internalization is an open question that needs to be clarified in empirical research. As such, the word "internalization" is a mere metaphor, and perhaps not even a very use-

ful one, insofar as it evokes a problematic or even misleading dualism of inside and outside. There are, however, very fruitful empirical studies on this process of internalization, which in particular clearly show the constitutive role of material practices. Current psychological research shows that mental arithmetic does indeed reflect material counting and calculation techniques and thus differs across different cultural contexts such as China and Western countries (Tang et al. 2006; Klein et al. 2011; Krajcsi and Szabó 2012).

§4.2.8. As Damerow explains, mental constructs obtained through first-order representation can again be represented by symbols, leading to second-order representations, defined as follows: "Second and higher order representations are representations of mental objects by symbols and symbol transformation rules which correspond to mental operations belonging to the cognitive structures constituting the mental objects." (Damerow 1996a, 374) Examples of second-order representations are nonconstructive numerical signs (such as Arabic numerals "1, 2, 3, ...") or the word "number" as a representation of the number concept. If we bear in mind that the cognitive structures mirror firstorder representations, we understand the meaning of the term "second-order representations": second-order representations symbolically represent mental objects which are based on first-order representations, and mutatis mutandis for higherorder representations. Higher-order representations show exactly the same structure as firstorder representations.

**§4.2.9.** Higher order-representations, as material artefacts, can be qualitatively novel tools (we will deal with such a case in the next subsection). But one should note that this is not necessarily the case, and the economy in the use of resources will generally push in the opposite direction, i.e. towards re-use of existing resources. Damerow describes such a case:

In the process of cognitive development, first order representations may change their function and turn into second order representations. The sequence of number words, for instance, develops from a first order representation of an ordinal structure into a second order representation of cardinality by means of correspondence relations constructed by the counting activity, and finally into a higher order representation of the concept of number. (Damerow

#### **1996a**, 375)

**§4.2.10.** Another striking example can be found in Overmann's analysis of Polynesian counting systems, where the same fruits play the role not only of the objects to be counted, but also of the counting device within an increasing exponential register (iterated routines of bundling first the fruits, then bundles of fruits, ...) (Overmann 2021a, 303–5).

**§4.2.11.** If we now return to our initial example of counting, we will better understand to what extent we are dealing here with a theory of "material" abstraction in the sense of a non-mental process and what exactly it means to call something "abstract." Essential to the emergence of numbers is a relation of representation, which consists of the fact that certain concrete material things are assigned other material things as substitutes or first-order representations. These symbols - tokens - are, on the one hand, themselves material, but they differ from the represented objects in other respects. They are easier to handle and can be rearranged. They therefore materially embody the cardinality of the represented objects in the sense of a potential abstraction. At the end of the story, they will be numbers. This allows an interesting comparison with usual logical reconstructions of the concept of numbers. Bertrand Russell, drawing on Giuseppe Peano's "definitions by abstraction," understood numbers as equivalence classes of classes of equal cardinality (Russell 1903; Peano 1901). This was meant to spell out the idea that a number is what classes of the corresponding cardinality have in common. Three sheep and three cows do indeed have the cardinality "three" in common. But their comparison does not trigger an abstraction because both classes behave physically similarly. According to Damerow, the situation is different with three sheep and three clay tokens which represent the sheep. Because the tokens are physically (and behaviorally) simpler, they can embody the cardinality.

**§4.2.12.** One can compare this with a similar case from linguistics. The sentences "Es regnet" and "It is raining" have the same meaning. So they have this meaning in common with each other. But their frictionless reciprocal translation does not embody a potential abstraction. It is different when the sentence "It is raining" is transferred from the spoken language to written language, i.e. from a verbal utterance to a written sentence. On the one hand, this is a mere change of medium (as in the previous case, a mere change of language of

guage). But this second change represents a material abstraction, as explained by the linguist Roy Harris, who spoke of an "autoglottic abstraction": "Utterances are automatically sponsored by those who utter them, even if they merely repeat what has been said before. Sentences, by contrast, have no sponsors: they are autoglottic abstractions." (Harris 1989, 104) The practice of writing detaches - or materially abstracts - the statements from the oral context in which they were automatically coupled with the act of endorsing. The material abstraction of the written sentence confronts us for the first time with a mere statement that can be considered without having to be asserted at the same time, which is a condition for the application of logical argumentation and proof.

§4.2.13. Damerow apparently assumed that with this mechanism of abstraction by representation he had found a way to reconstruct the entire development of mathematics. As examples of the results of higher-order representations, he cites the generalization of the concept of number to negative, irrational, and imaginary numbers, the replacement of numbers by variables, and also the development of meta-vocabulary such as the concept of "number" itself. For our purposes, however, it is important that even the lowest points of such abstractions, namely simple numbers, appear in the list of examples, "abstract arithmetic with number signs independent of concrete applications" (Damerow 1994, 264). The idea underlying this notion of the genesis of numbers can thus be roughly summarized as follows: The story begins with concrete objects of practical life, e.g., sheep. Then tokens are added, which are also concrete objects but have a representative function. The tokens give rise to systems of numerals, which, however, are not yet "abstract" numbers and are only used in practical contexts, namely for administrative purposes and, at the very most, in a speculative manner in the training of administrative officials. In this context, the numerical signs gradually develop a life of their own and become representatives of numbers. Now the signs, which were originally mere tools of bookkeeping and economic administration, represent mathematical objects in their own right, and only then do numbers "exist."

#### §4.3. From tokens to abstract counting

**§4.3.1.** This is how far Damerow got. He and his colleagues deciphered the rules of use of Mesopotamian number signs; he studied intensively non-European techniques of counting, such

as those of Papua New Guinea and set up models to describe the development of numerical thinking and numbers from the earliest beginnings through the mathematics of ancient Greece to modern mathematics. We can now ask ourselves whether, within the framework of this approach, we can also begin to understand the emergence of "abstract" counting in the sense of counting without counting anything in particular. Damerow has led us to the idea that tokens are, so to speak, abstract objects, that is, they embody a material abstraction. But the tokens, as counting tools, always remain bound to concrete objects that are counted. Admittedly, this connection to the objects loses force as soon as it becomes clear that it is actually based on a convention and that other kinds of things can also be counted with the same tokens. But it must always be real things that are counted. Moving on to "pure" counting does not make sense at this point, which raises the question of another material abstraction mechanism.

§4.3.2. We can take an important hint from the work of Denise Schmandt-Besserat, who suggested that the Sumerian numeral signs have a predecessor in clay bullae, which, in addition to the tokens contained in them, additionally show the externally impressed marks of these tokens. The production of these externally marked clay bullae has been described by Schmand-Besserat: "the tokens themselves were pressed into the soft clay before being enclosed in the envelope" (Schmandt-Besserat 1980, 383). Also Nissen, Englund and Damerow refer to this technique (cf. also Damerow and Meinzer 1995, 20–33). They count the "striking correspondence between type and number of counters and the signs impressed in the surface of the clay balls" as

the strongest evidence for the proposed function of the tokens contained inside the bullae as counters, because exact number was obviously important. In addition, the impressed signs on the surface of the bullae are very similar to those observed on numerical tablets, and the circumstances under which some of the artifacts have been found point unambiguously to a similar function of both symbol types. As far as the latter are concerned, it is clear the numerical signs on archaic tablets developed from such signs. (Nissen, Damerow, and Englund 1993, 127)

§4.3.3. There is a consensus here that the tokens

evolved (via the impressed marks) into the later number signs of the numerical tablets. But even if there exists a material continuity, leading from tokens via impressed marks to Cuneiform numerals on tablets, the question arises of how to decipher it from the point of view of the *meaning* of the counting devices and symbols used and how to identify the abstractions involved in this evolutionary story. Schmandt-Besserat's reflections on the marked clay bullae hold a surprise here. She writes (Schmandt-Besserat 1980, 384): "The process from token to writing amounted to bringing the symbolism of the tokens to a second degree of abstraction by reducing the objects to a two-dimensional sign." Admittedly, Schmandt-Besserat seems to use the term *abstraction* here in the sense common in the history of art (as also Overmann 2021b, 187, notes), i.e. as a contrast term to "realistic" or "naturalistic representation": a representation is the more abstract the less it resembles the represented object. The idea of a "second degree of abstraction" nevertheless is intriguing and raises the question whether it is possible to take this expression in the strong epistemological sense, namely as the production of cognitive content?

**§4.3.4.** In fact, we can spell out the idea of a "second degree of abstraction" in the terms of Damerow's model. The tokens in the clay bullae provide an ordinary first-order representation embodying a material abstraction. If the tokens are now pressed into the surface of the soft clay before being enclosed in the bulla, another representation is created. This is again a first-order representation (marks as representatives of tokens) *of another first-order representation* (tokens as representatives of e.g., sheep).

**§4.3.5.** In this sense, we could call it a secondorder representation, but this would lead to confusion, since Damerow meant something else by this term. For Damerow, second-order representations were not representations of first-order representations, but rather representations of the mental constructs that are themselves based on first-order representations, which then are "externally represented in tools of arithmetical thinking and in second-order concepts related to their use." (Damerow 1996b, 140) The marks on the clay bullae are therefore not a second-order representation in the sense of Damerow, but the second link of a two-part chain of first-order representations. But let us now look at the consequences of this chain of representations.

§4.3.6. In our view, this interpretation of the impressed marks as second parts of a two-part chain of first order representations could close the gap between tokens (as symbols for objects) and numerals (as symbols for numbers) or between counting *something* and counting *tout court*. The crucial point is that the impressed marks do not stand for *objects* in the sense of goods to be counted; they stand for tokens which also happen to stand for objects of a certain category but could also stand for things of a different kind. As a matter of fact, the marks were produced as (indirect) representations of concrete things, but the mechanism of twofold representation erased the link to them. Counting at the level of marks (second first-order representations) thus ceases to be counting of concrete things, without, however, provoking the objection that this makes no sense, because the marks still referred to the tokens (first first-order representations); however, the meaning of the tokens becomes indeterminate. The role of the tokens in the mechanism of abstraction through two-part first order representation can be compared to that of 'parameters' in the logical calculus of natural deduction, which are also called 'ambiguous names' or 'temporary constants' (Suppes 1957, 81). Whereas 'variables' can take any value and 'constants' have a determinate value, parameters stand for a fixed, but indeterminate, arbitrary or unknown value. In everyday use, the tokens have a definite meaning, they stand for definite objects such as sheep or oil amphorae. In pure counting, the units have become variables that can stand for anything. In the two-part first-order representation, we have reached an intermediate state in which the impressed marks, through the tokens, still stand for something determinate, but where it has become basically irrelevant what exactly they stand for.

**§4.3.7.** If we accept this idea, we see very quickly how the story can continue from here. The next question to be asked is what potential abstraction is contained in the impressed tokens. One cannot simply repeat Damerow's analysis of tokens here. Tokens, after all, were characterized by the fact that they were much easier to handle than the things counted (e.g. sheep). In comparison, impressed tokens have very different physical properties. For example, they have a fixed spatial order that can no longer be changed. Damerow and Englund, however, also noticed precisely this difference between the manipulation of marks and

the manipulation of tokens and attached an interesting consideration to it:

Figure 115 [referring to P003535] displays a tablet representing the most simple case of such an 'addition.' The entries on the obverse contain the sign  $\square$  a total of seven times, each sign representing a jar filled with beer. On the reverse side of the tablet, the seven signs reappear, forming together with the sign for 'jar' the 'total' of the registered jars. Obviously, this procedure of addition by sign repetition hardly differs from the proto-arithmetical method of totaling counters by placing them next to each other. Originally, therefore, addition was not an arithmetical operation in the proper sense of the word at all. It was much the same as the manipulation of the registered goods themselves. The essential innovation brought about by the invention of writing may be seen in the first place in the fact that the summanda did not disappear through the totaling, but were retained simultaneously next to the total as separate units of information, whereas in the totaling of tokens the same objects that first represent the *summanda* also form the sum. This structural difference between the symbolic operation of the addition and the represented real process of summarizing probably formed the point of departure for later abstraction of object quantities, finally leading to an abstract concept of number. (Nissen, Damerow, and Englund 1993, 132, emphasis added)

**§4.3.8.** This observation is intriguing because it shows that from an abstraction-theoretical point of view the impressed marks do not represent a step backwards from the tokens, but rather offer new, unexpected possibilities for abstractions in new directions. It may even be correct to say that with this abstracting of an "abstract concept of number," the first-order representation of a first-order representation. The individual steps of this chain of abstractions, as we have traced them so far, are therefore:

- Concrete things no representation no abstraction;
- 2. Tokens 1<sup>st</sup> order representation concrete counting;
- 3. Impressed marks on bullae 1<sup>st</sup> order representation of a 1<sup>st</sup> order representation ab-

stract counting;

4. Numerals on tablets – 2<sup>nd</sup> order representation – abstract concept of number.

**§4.3.9.** Although we tell a story which differs from Malafouris' account of the transition from tokens to impressed marks (Malafouris focuses on the newly emerging properties of indexicality and iconicity), we come to the very same conclusion: the reconstruction shows how "a difficult and inherently meaningless conceptual problem (counting) [...] as an embodied and mediated act became meaningful" (Malafouris 2013, 114-5) Overall, we see a story in which, still in Malafouris' words: "the clay token does not stand for or represent the concept of number but instead brings forth the concept of number." (Malafouris 2010, 40) The kind of abstraction we are dealing with here might be called "material abstraction" in Overmann's terms (Overmann 2021b, 223, and Overmann 2021a, 316), to the extent that the term "material" signals that we are not dealing with a mental process. But "material" should not be understood as "mechanistic" or "physicalist." Though the abstraction is a non-intentional by-product, it occurs within - and only within - the realm of human action. "Material abstraction" then means that certain human artefacts are used in novel way and that they inscribe themselves into new social forms of action, thus unfolding new meanings. Marks of tokens are still material objects, but they are used in a way which legitimizes our saying that they became numerals, i.e., symbols of "abstract numbers," with the term "abstract" referring to the way the material symbols are used. It is the way they are used that makes them abstract, as Mattessich once put it in a Wittgensteinian spirit (Mattessich 1987, 79). This again demonstrates that "abstract" does not mean "mental," and that numbers can thus be described as material and abstract at the same time without contradiction. This should dispel Overmann's reservations about the term "abstract." Numbers have indeed been created out of clay with our hands – not just with our minds. They are material artefacts. It is their particular use, the role they play in a social form of practice, that makes them abstract without their ceasing to be material. The process of abstraction as such is not merely intentional or merely mental. But it can provoke reflection and the invention of new metavocabulary, such as the term "number."

**§4.3.10.** The story we propose here fits well into Graeber and Wengrow's recent account of the Neolithic revolution as a "media revolution", that is, the discovery of clay as a new basis for the production of food, shelter, and abstract thought:

Recall that flood-retreat farming required people to establish durable settlements in mud-based environments, like swamps and lake margins. Doing so meant becoming intimate with the properties of soils and clays, carefully observing their fertility under different conditions, but also experimenting with them as tectonic materials, or even as vehicles of abstract thought. As well as supporting new forms of cultivation, soil and clay - mixed with wheat and chaff - became basic materials of construction: essential in building the first permanent houses; used to make ovens, furniture and insulation - almost everything, in fact, except pottery, a later invention in this part of the world [i.e. the Fertile Crescent]. (Graeber and Wengrow 2021, 240)

However, if we consider clay not merely as a "vehicle" of abstract thought that exists independently of it, we can go a step further and understand objects made of clay as an affordance or potential means to thinking new thoughts. In this sense, it is not meant metaphorically when we say that the numbers were created from clay.

§4.3.11. One problem with this kind of account, of course, is that we cannot attach any kind of necessity to the mechanisms described. Did the concatenation of repeated first-order representations really lead to a culturally validated abstraction - or did the Mesopotamian scribes simply forget the strange origin of their numeral signs in the impressed marks of the clay bullae after two or three generations, without this circumstance's leading to any novel abstraction? Perhaps this problem is not a weakness of our hypothesis, but a consequence of Historical Epistemology - as opposed to Genetic Epistemology.<sup>2</sup> If we take Historical Epistemology seriously, we perhaps have to pay the price that there is no longer any developmental determinism in the history of cognition, its forms and contents, but that everything turns out to be historically contingent. If this is so, we can at least claim that with the consider-

<sup>&</sup>lt;sup>2</sup> On the nature of abstraction and representation in general, see Renn 2020, Ch. 3; and on the relation of genetic to historical epistemology in particular, pp. 57–62.

ations from this section we have shown how it might have been, and thus in turn to have shown that first, the history of abstractions can be told as a material rather than a mental history, and second, that Damerow's approach is still helpful and fruitful in this kind of endeavor.

#### §5. Conclusions

**§5.1.** Let us come back to Overmann's critique of Damerow. It has become clear in our analysis that Damerow did not adhere to an account according to which "abstract numbers" emerged from a technique of "concrete counting" in which "tokens were confused with the objects enumerated" or in which "tokens represented concrete numbers", as Overmann put it (Overmann 2018, 5). He clearly did not believe that ontogenetic maturation is the mechanism of change in social thinking, and, above all, he neither ignored the materiality of writing nor denied that even elaborate numbers depend on material forms. In our view, Malafouris' account does much more justice to Damerow:

Damerow argued that the initial emergence of the concept of conservation of quantity is tied to the substantive reality and concrete use of clay tokens and not to any preexisting cognitive skills of an arithmetical nature. Moreover, he contends that the physical qualities of the material sign as well as the forms of social interaction mediated by those signs influence this process by marking the horizon of possibilities for their ontogenetic realization. [...He] asserts that the real impetus behind this transition to proto-arithmetic operations comes from the change in the medium of representation (i.e. clay tablets) and the social conditions that surround it, and not from any antecedent change in cognitive structure. (Malafouris 2013, 111)

**§5.2.** Although the grounds for any particular cognitive development in history are to be determined empirically, it is clear that Damerow was more likely to look for the causes of changes in cognitive structures in changes in human practices, than the other way round. The lesson to be learned from Damerow's work is that the material means of representation and the practices associated with them are the determining factors in cognitive development.

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