

## Two.II Linear Independence

*Linear Algebra*, edition four

Jim Hefferon

<http://joshua.smcvt.edu/linearalgebra>

## Definition and examples

## Linear independence

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Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \cdots + c_n \vec{s}_n$$

visually sets off  $\vec{s}_0$ , algebraically there is nothing special about that vector in that equation. For any  $\vec{s}_i$  with a coefficient  $c_i$  that is non-0 we can rewrite to isolate  $\vec{s}_i$ .

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that  $\vec{s}_0, \vec{s}_1, \dots, \vec{s}_n$  are in a *linear relationship* and put all of the vectors on the same side.

1.5 *Lemma*    A subset  $S$  of a vector space is linearly independent if and only if among its elements the only linear relationship  $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$  is the trivial one,  $c_1 = 0, \dots, c_n = 0$  (where  $\vec{s}_i \neq \vec{s}_j$  when  $i \neq j$ ) .

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*Proof* If  $S$  is linearly independent then no vector  $\vec{s}_i$  is a linear combination of other vectors from  $S$  so there is no linear relationship where some of the  $\vec{s}$ 's have nonzero coefficients.

If  $S$  is not linearly independent then some  $\vec{s}_i$  is a linear combination  $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$  of other vectors from  $S$ . Subtracting  $\vec{s}_i$  from both sides gives a relationship involving a nonzero coefficient, the  $-1$  in front of  $\vec{s}_i$ . QED

So to decide if a list of vectors  $\vec{s}_0, \dots, \vec{s}_n$  is linearly independent, set up the equation  $\vec{0} = c_0\vec{s}_0 + \cdots + c_n\vec{s}_n$ , and calculate whether it has any solutions, other than the trivial one where all coefficients are zero.

*Example* This set of vectors in the plane  $\mathbb{R}^2$  is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The only solution to this equation

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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*Example* In the vector space of cubic polynomials

$\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$  the set  $\{1 - x, 1 + x\}$  is linearly independent. Setting up the equation  $c_0(1 - x) + c_1(1 + x) = 0$  and considering the constant term and linear term, leads to this system

$$\begin{aligned} c_0 + c_1 &= 0 \\ -c_0 + c_1 &= 0 \end{aligned}$$

which has only the trivial solution.



*Example* The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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*Example* This subset of  $\mathbb{R}^3$  is linearly dependent.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$$

One way to see that is to spot that the third vector is twice the first plus the second. Another way is to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 3c_1 + 6c_3 &= 0 \end{aligned}$$

and note that it has more than just the solution  $c_1 = c_2 = c_3 = 0$ .

1.2 *Lemma*    Where  $V$  is a vector space,  $S$  is a subset of that space, and  $\vec{v}$  is an element of that space,  $[S \cup \{\vec{v}\}] = [S]$  if and only if  $\vec{v} \in [S]$ .

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*Proof* Half of the if and only if is immediate: if  $\vec{v} \notin [S]$  then the sets are not equal because  $\vec{v} \in [S \cup \{\vec{v}\}]$ .

For the other half assume that  $\vec{v} \in [S]$  so that  $\vec{v} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n$  for some scalars  $c_i$  and vectors  $\vec{s}_i \in S$ . We will use mutual containment to show that the sets  $[S \cup \{\vec{v}\}]$  and  $[S]$  are equal. The containment  $[S \cup \{\vec{v}\}] \supseteq [S]$  is clear.

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To show containment in the other direction let  $\vec{w}$  be an element of  $[S \cup \{\vec{v}\}]$ . Then  $\vec{w}$  is a linear combination of elements of  $S \cup \{\vec{v}\}$ , which we can write as  $\vec{w} = c_{n+1} \vec{s}_{n+1} + \cdots + c_{n+k} \vec{s}_{n+k} + c_{n+k+1} \vec{v}$ . (Possibly some of the  $\vec{s}_i$ 's from  $\vec{w}$ 's equation are the same as some of those from  $\vec{v}$ 's equation but that does not matter.) Expand  $\vec{v}$ .

$$\vec{w} = c_{n+1} \vec{s}_{n+1} + \cdots + c_{n+k} \vec{s}_{n+k} + c_{n+k+1} \cdot (c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n)$$

Recognize the right hand side as a linear combination of linear combinations of vectors from  $S$ . Thus  $\vec{w} \in [S]$ .

QED

1.3 *Corollary* For  $\vec{v} \in S$ , omitting that vector does not shrink the span  $[S] = [S - \{\vec{v}\}]$  if and only if it is dependent on other vectors in the set  $\vec{v} \in [S]$ .

*Example* These two subsets of  $\mathbb{R}^3$  have the same span

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

because in the first set  $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$ .

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1.14 *Corollary* A set  $S$  is linearly independent if and only if for any  $\vec{v} \in S$ , its removal shrinks the span  $[S - \{\vec{v}\}] \subsetneq [S]$ .

*Proof* This follows from Corollary 1.3. If  $S$  is linearly independent then none of its vectors is dependent on the other elements, so removal of any vector will shrink the span. If  $S$  is not linearly independent then it contains a vector that is dependent on other elements of the set, and removal of that vector will not shrink the span. QED

1.15 *Lemma* Suppose that  $S$  is linearly independent and that  $\vec{v} \notin S$ . Then the set  $S \cup \{\vec{v}\}$  is linearly independent if and only if  $\vec{v} \notin [S]$ .



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*Proof* We will show that  $S \cup \{\vec{v}\}$  is not linearly independent if and only if  $\vec{v} \in [S]$ .

Suppose first that  $\vec{v} \in [S]$ . Express  $\vec{v}$  as a combination  $\vec{v} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n$ . Rewrite that  $\vec{0} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n - 1 \cdot \vec{v}$ . Since  $\vec{v} \notin S$ , it does not equal any of the  $\vec{s}_i$  so this is a nontrivial linear dependence among the elements of  $S \cup \{\vec{v}\}$ . Thus that set is not linearly independent.

Now suppose that  $S \cup \{\vec{v}\}$  is not linearly independent and consider a nontrivial dependence among its members  $\vec{0} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n + c_{n+1} \cdot \vec{v}$ . If  $c_{n+1} = 0$  then that is a dependence among the elements of  $S$ , but we are assuming that  $S$  is independent, so  $c_{n+1} \neq 0$ . Rewrite the equation as  $\vec{v} = (c_1/c_{n+1})\vec{s}_1 + \cdots + (c_n/c_{n+1})\vec{s}_n$  to get  $\vec{v} \in [S]$  QED

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*Proof* If  $S = \{\vec{s}_1, \dots, \vec{s}_n\}$  is linearly independent then  $S$  itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent,  $S$  contains a vector  $\vec{v}_1$  that is a linear combination of the others. Define the set  $S_1 = S - \{\vec{v}_1\}$ . By Corollary 1.3 the span does not shrink  $[S_1] = [S]$ .

If  $S_1$  is linearly independent then we are done. Otherwise iterate: take a vector  $\vec{v}_2$  that is a linear combination of other members of  $S_1$  and discard it to derive  $S_2 = S_1 - \{\vec{v}_2\}$  such that  $[S_2] = [S_1]$ . Repeat this until a linearly independent set  $S_j$  appears; one must appear eventually because  $S$  is finite and the empty set is linearly independent. QED

*Example* Consider this subset of  $\mathbb{R}^2$ .

$$S = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5\} = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

The linear relationship

$$r_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} + r_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + r_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + r_5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives a system of equations.

$$\begin{array}{rclcl} 2r_1 + 3r_2 + r_3 & + & r_5 & = & 0 \\ 2r_1 + 3r_2 + 4r_3 - r_4 - r_5 & = & 0 \end{array} \xrightarrow{-\rho_1 + \rho_2} \begin{array}{rclcl} 2r_1 + 3r_2 + r_3 & + & r_5 & = & 0 \\ & + & 3r_3 - r_4 - 2r_5 & = & 0 \end{array}$$

Parametrize by expressing the leading variables  $r_1$  and  $r_3$  in terms of the free variables.

$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R} \right\}$$

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Set  $r_5 = 1$  and  $r_2 = r_4 = 0$  to get  $r_1 = -5/6$  and  $r_3 = 2/3$ ,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that  $\vec{s}_5$  is in the span of the set  $\{\vec{s}_1, \vec{s}_3\}$ .

Similarly, setting  $r_4 = 1$  and the other parameters to 0 shows  $\vec{s}_4$  is in the span of the set  $\{\vec{s}_1, \vec{s}_3\}$ . Also, setting  $r_2 = 1$  and the other parameters to 0 shows  $\vec{s}_2$  is in the span of the same set.

So without shrinking the span we can omit the vectors  $\vec{s}_2, \vec{s}_4, \vec{s}_5$  associated with the free variables. The set of vectors associated with the leading variables,  $\{\vec{s}_1, \vec{s}_3\}$ , is linearly independent and so we cannot omit any members without shrinking the span.

1.19 *Corollary*     A subset  $S = \{\vec{s}_1, \dots, \vec{s}_n\}$  of a vector space is linearly dependent if and only if some  $\vec{s}_i$  is a linear combination of the vectors  $\vec{s}_1, \dots, \vec{s}_{i-1}$  listed before it.

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*Proof* Consider  $S_0 = \{\}$ ,  $S_1 = \{\vec{s}_1\}$ ,  $S_2 = \{\vec{s}_1, \vec{s}_2\}$ , etc. Some index  $i \geq 1$  is the first one with  $S_{i-1} \cup \{\vec{s}_i\}$  linearly dependent, and there  $\vec{s}_i \in [S_{i-1}]$ .

QED



## Linear independence and subset

1.20 *Lemma* Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

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This table summarizes the cases.

	$\hat{S} \subset S$	$\hat{S} \supset S$
$S$ independent	$\hat{S}$ must be independent	$\hat{S}$ may be either
$S$ dependent	$\hat{S}$ may be either	$\hat{S}$ must be dependent

An example of the lower left is that the set  $S$  of all vectors in the space  $\mathbb{R}^2$  is linearly dependent but the subset  $\hat{S}$  consisting of only the unit vector on the  $x$ -axis is independent. By interchanging  $\hat{S}$  with  $S$  that's also an example of the upper right.