

## Two.II Linear Independence

*Linear Algebra*, edition four

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Definition and examples

## Linear independence

- 1.4 *Definition* In any vector space, a set of vectors is *linearly independent* if none of its elements is a linear combination of the others from the set. Otherwise the set is *linearly dependent*.

Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \cdots + c_n \vec{s}_n$$

visually sets off  $\vec{s}_0$ , algebraically there is nothing special about that vector in that equation. For any  $\vec{s}_i$  with a coefficient  $c_i$  that is non-0 we can rewrite to isolate  $\vec{s}_i$ .

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that  $\vec{s}_0, \vec{s}_1, \dots, \vec{s}_n$  are in a *linear relationship* and put all of the vectors on the same side.

1.5 *Lemma* A subset  $S$  of a vector space is linearly independent if and only if among its elements the only linear relationship  $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$  is the trivial one,  $c_1 = 0, \dots, c_n = 0$  (where  $\vec{s}_i \neq \vec{s}_j$  when  $i \neq j$ ).

*Proof* If  $S$  is linearly independent then no vector  $\vec{s}_i$  is a linear combination of other vectors from  $S$  so there is no linear relationship where some of the  $\vec{s}$ 's have nonzero coefficients.

If  $S$  is not linearly independent then some  $\vec{s}_i$  is a linear combination  $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$  of other vectors from  $S$ . Subtracting  $\vec{s}_i$  from both sides gives a relationship involving a nonzero coefficient, the  $-1$  in front of  $\vec{s}_i$ . QED

So to decide if a list of vectors  $\vec{s}_0, \dots, \vec{s}_n$  is linearly independent, set up the equation  $\vec{0} = c_0\vec{s}_0 + \cdots + c_n\vec{s}_n$ , and calculate whether it has any solutions, other than the trivial one where all coefficients are zero.

*Example* This set of vectors in the plane  $\mathbb{R}^2$  is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The only solution to this equation

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is trivial  $c_1 = 0, c_2 = 0$ .

*Example* In the vector space of cubic polynomials

$\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$  the set  $\{1 - x, 1 + x\}$  is linearly independent. Setting up the equation  $c_0(1 - x) + c_1(1 + x) = 0$  and considering the constant term and linear term, leads to this system

$$\begin{aligned} c_0 + c_1 &= 0 \\ -c_0 + c_1 &= 0 \end{aligned}$$

which has only the trivial solution.

*Example* The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We showed in Lemma One.III.2.5 that in any echelon form matrix the nonzero rows form a linearly independent set.

*Example* This subset of  $\mathbb{R}^3$  is linearly dependent.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$$

One way to see that is to spot that the third vector is twice the first plus the second. Another way is to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 3c_1 + 6c_3 &= 0 \end{aligned}$$

and note that it has more than just the solution  $c_1 = c_2 = c_3 = 0$ .

1.2 *Lemma* Where  $V$  is a vector space,  $S$  is a subset of that space, and  $\vec{v}$  is an element of that space,  $[S \cup \{\vec{v}\}] = [S]$  if and only if  $\vec{v} \in [S]$ .

*Example* The book has the proof; here is an illustration. The span of this set is the  $xy$ -plane.

$$P = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$$

If we expand the set by adding a vector  $\{\vec{p}_1, \vec{p}_2, \vec{q}\}$  then there are two possibilities.

$$P_0 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\} \quad P_1 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

If the new vector is already in the starting span  $\vec{q} \in [P]$  then the span is unchanged  $[P_0] = [P]$ . But if the new vector is outside the starting span  $\vec{q} \notin [P]$  then the span grows  $[P_1] \supsetneq [P]$ .

1.3 *Corollary* For  $\vec{v} \in S$ , omitting that vector does not shrink the span  $[S] = [S - \{\vec{v}\}]$  if and only if it is dependent on other vectors in the set  $\vec{v} \in [S]$ .

*Example* These two subsets of  $\mathbb{R}^3$  have the same span

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

because in the first set  $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$ .

1.14 *Corollary* A set  $S$  is linearly independent if and only if for any  $\vec{v} \in S$ , its removal shrinks the span  $[S - \{\vec{v}\}] \subsetneq [S]$ .

*Example* This is a linearly independent subset of  $\mathcal{P}_3$

$$S = \{1 + x, 1 - x, x^2\}$$

Removal of any element, such as if we remove  $1 - x$  to get  $\hat{S} = \{1 + x, x^2\}$ , will make the span smaller:  $[\hat{S}] \subsetneq [S]$ .



1.15 *Lemma* Suppose that  $S$  is linearly independent and that  $\vec{v} \notin S$ . Then the set  $S \cup \{\vec{v}\}$  is linearly independent if and only if  $\vec{v} \notin [S]$ .

*Example* The book has the proof; here is an illustration. Consider this linearly independent subset of  $\mathcal{P}_2$ .

$$S = \{1 - x, 1 + x\}$$

Its span  $[S]$  is the set of linear polynomials  $\{a + bx \mid a, b \in \mathbb{R}\}$ . (To check: consider  $a + bx = r_1(1 - x) + r_2(1 + x)$ , which gives a linear system with equations  $r_1 + r_2 = a$  and  $-r_1 + r_2 = b$ , having the solution  $r_2 = (1/2)a + (1/2)b$  and  $r_1 = (1/2)a - (1/2)b$ .)

Here are two supersets.

$$S_1 = S \cup \{2 + 2x\} \quad S_2 = S \cup \{2 + x^2\}$$

On the left, adding a linear polynomial just adds “repeat information” so  $[S_1] = [S]$  and  $S_1$  is linearly dependent.

The right, with “new information,” enlarges the span  $[S_2] = \mathcal{P}_2 \supsetneq [S]$  and the new set  $S_2$  is also linearly independent. (To check this, use  $a + bx + cx^2 = r_1(1 - x) + r_2(1 + x) + r_3(2 + x^2)$  to get a linear system with solution  $r_3 = c$ ,  $r_2 = (1/2)a + (1/2)b$  and  $r_1 = (1/2)a - (1/2)b - c$ .)

1.17 *Corollary* In a vector space, any finite set has a linearly independent subset with the same span.

The book has a proof. Instead, consider the example on the next slide.

*Example* Consider this subset of  $\mathbb{R}^2$ .

$$S = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5\} = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

The linear relationship

$$r_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} + r_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + r_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + r_5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives a system of equations.

$$\begin{array}{rclcl} 2r_1 + 3r_2 + r_3 & & + r_5 = 0 & \xrightarrow{-\rho_1 + \rho_2} & 2r_1 + 3r_2 + r_3 & + r_5 = 0 \\ 2r_1 + 3r_2 + 4r_3 - r_4 - r_5 = 0 & & & & & + 3r_3 - r_4 - 2r_5 = 0 \end{array}$$

Parametrize by expressing the leading variables  $r_1$  and  $r_3$  in terms of the free variables.

$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R} \right\}$$

Set  $r_5 = 1$  and  $r_2 = r_4 = 0$  to get  $r_1 = -5/6$  and  $r_3 = 2/3$ ,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that  $\vec{s}_5$  is in the span of the set  $\{\vec{s}_1, \vec{s}_3\}$ .

Similarly, setting  $r_4 = 1$  and the other parameters to 0 shows  $\vec{s}_4$  is in the span of the set  $\{\vec{s}_1, \vec{s}_3\}$ . Also, setting  $r_2 = 1$  and the other parameters to 0 shows  $\vec{s}_2$  is in the span of the same set.

So without shrinking the span we can omit the vectors  $\vec{s}_2, \vec{s}_4, \vec{s}_5$  associated with the free variables. The set of vectors associated with the leading variables,  $\{\vec{s}_1, \vec{s}_3\}$ , is linearly independent and so we cannot omit any members without shrinking the span.

1.19 *Corollary* A subset  $S = \{\vec{s}_1, \dots, \vec{s}_n\}$  of a vector space is linearly dependent if and only if some  $\vec{s}_i$  is a linear combination of the vectors  $\vec{s}_1, \dots, \vec{s}_{i-1}$  listed before it.

*Proof* Consider  $S_0 = \{\}$ ,  $S_1 = \{\vec{s}_1\}$ ,  $S_2 = \{\vec{s}_1, \vec{s}_2\}$ , etc. Some index  $i \geq 1$  is the first one with  $S_{i-1} \cup \{\vec{s}_i\}$  linearly dependent, and there  $\vec{s}_i \in [S_{i-1}]$ .

QED

## Linear independence and subset

1.20 *Lemma* Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

*Proof* Both are clear.

QED

This table summarizes the cases.

	$\hat{S} \subset S$	$\hat{S} \supset S$
$S$ independent	$\hat{S}$ must be independent	$\hat{S}$ may be either
$S$ dependent	$\hat{S}$ may be either	$\hat{S}$ must be dependent

An example of the lower left is that the set  $S$  of all vectors in the space  $\mathbb{R}^2$  is linearly dependent but the subset  $\hat{S}$  consisting of only the unit vector on the  $x$ -axis is independent. By interchanging  $\hat{S}$  with  $S$  that's also an example of the upper right.