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The Algebraic Structure of Sets of Regions

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Abstract. The provision of ontologies for spatial entities is an important topic in spatial information theory. Heyting algebras, co-Heyting algebras, and bi-Heyting algebras are structures having considerable potential for the theoretical basis of these ontologies. This paper gives an introduction to these Heyting structures, and provides evidence of their importance as algebraic theories of sets of regions. The main evidence is a proof that elements of certain Heyting algebras provide models of the Region-Connection Calculus developed by Cohn et al. By using the mathematically well known techniques of "pointless topology", it is straightforward to conduct this proof without any need to assume that regions consist of sets of points. Further evidence is provided by a new qualitative theory of regions with indeterminate boundaries. This theory uses modal operators which are related to the algebraic operations present in a bi-Heyting algebra.

1 Introduction

It has been recognized that the relationships of parthood, contact and boundary form fundamental components of an ontology of regions in space. Johnson [Joh87] goes further, writing that *"Much of the structure, value and purposeness we take for granted as built into our world consists chiefly of interwoven and superimposed schemata."* He goes on to provide a list of "image schemata", which includes part-whole, centre-periphery and contact.

Three significant formal ontologies of space have been provided by Cohn et al. [GGC96], by Egenhofer and Franzosa [FE91,FE95], and by Smith [Smi96]. Cohn et al. have developed a theory of regions and their relationships, based upon a single primitive contact relation, called connection, between regions. Egenhofer and Franzosa provide their well known model of binary topological relations between point sets. Smith introduces a mix of mereology and topology to provide a theory of parthood and boundary for spatial entities.

The general contention of this paper is that Heyting algebras, and related structures, provide elegant and natural theories of parthood and boundary which have close connections to the above three ontologies. Heyting algebras are well established algebraic structures [Joh82,Vic89], but the dual notion of a co-Heyting algebra [Law86,Law91] is less well-known. A bi-Heyting algebra [RZ96] is both a

Heyting and a co-Heyting algebra. The formal definitions are given later in the paper, but in broad terms a co-Heyting algebra is a generalization of a Boolean algebra that allows a relaxation of the constraint that a proposition and its negation may not hold together. In terms of the ontology of regions, this is a useful relaxation, for being able to represent and reason about locations that are both in and not in the region at the same time gives us a much richer notion of boundary than can be obtained within a Boolean algebra.

We provide convincing evidence of the importance of Heyting algebras, by demonstrating how they can provide models of Cohn's Region-Connection Calculus. It is significant that in this approach the notion of a region as consisting of a set of points is not used at all. We are thus able to produce a pure framework in which the regional entities themselves are fundamental.

There is a great deal of interest in regions with indeterminate boundaries, see [BF96] for a collection of recent papers on this topic. The final section of the present paper considers a new approach to these regions, using concepts related to those used in the theory of bi-Heyting algebras. Modal operators and two forms of negation are applied to model notions such as the degree of certainty whether a location is in a region, and the *core* of a region, containing locations that are definitely members. Semantics are provided in terms of environments of locations, in which observations can be made concerning membership of the region. A major difficulty with fuzzy logic approaches to regions with indeterminate boundary is that qualitative notions must be translated into quantitative arguments involving numerical membership functions and certainty factors. The strength of the approach developed here is that the theory is qualitative throughout.

The overall structure of the paper is as follows. Section 2 introduces Boolean, Heyting, co-Heyting and bi-Heyting algebras, giving examples of each. In section 3 we discuss two examples of bi-Heyting algebras where the elements can be seen as regions. Section 4 shows how a Heyting algebra satisfying some regularity, completeness and connectedness conditions provides a model of the RCC formalism of Cohn and his colleagues. Section 5 uses some constructs from the theory of bi-Heyting algebras to provide a qualitative ontology of regions with indeterminate boundary. The paper describes the start that we have made in applying Heyting algebras to spatial ontologies, and the conclusion outlines our intentions for further work.

2 Heyting Algebras and co-Heyting Algebras

The aim of this section is to collect together the definitions of Heyting algebras, co-Heyting algebras, and bi-Heyting algebras. As our intention is to demonstrate the relevance of certain algebraic structures to spatial information theory, we have kept this section as short as possible. For more details of some of the mathematics relevant to Heyting algebras, the first three chapters of [Vic89] or the first ten pages of [Joh82] are recommended.

Definition 1. *A **lattice** is a set, A, equipped with elements, $0, 1 \in A$, and binary operations $\vee, \wedge : A \times A \to A$ satisfying the following conditions.*

1. *$\vee$ and $\wedge$ are associative, commutative and idempotent.*
2. *$x \vee 0 = x$ and $x \wedge 1 = x$ for all $x \in A$.*
3. *$x \wedge (x \vee y) = x = x \vee (x \wedge y)$ for all $x, y \in A$.*

The operations $\wedge$ and $\vee$ are usually called "meet" and "join" respectively. A lattice is said to be **distributive** if, for all $x, y, z \in A$, it satisfies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2. *A **Boolean algebra** is a distributive lattice, A, equipped with an operation, "negation", $\neg : A \to A$, such that $\neg x \wedge x = 0$ and $\neg x \vee x = 1$ for all $x \in A$.*

Mereology deals with parts of wholes and their relationships. An important example of the set of parts of a whole, which is a Boolean algebra, is provided by the powerset of any set, X, in which $\wedge$, $\vee$, $\neg$, 0 and 1 are intersection, union, complement, the empty set, and the set X itself. Another important example is provided by the formulae of propositional calculus for classical logic in which $\wedge$, $\vee$, and $\neg$ have the usual logical interpretation, and 0 and 1 are false and true respectively.

In any lattice we can define a partial order by $x \leq y$ iff $x \vee y = y$, or equivalently, iff $x \wedge y = x$. In the powerset case this partial order is is usual subset relation, $x \subseteq y$. In a lattice, $x \vee y$, is the least upper bound of the set $\{x, y\}$ with respect to $\leq$. Similarly, $x \wedge y$ is the greatest lower bound of $\{x, y\}$. A lattice having least upper bounds, and greatest lower bounds for all sets of elements, not just finite ones, is said to be **complete**.

Definition 3. *A **Heyting algebra** is a distributive lattice, A, equipped with a binary operation, "implication", $\Rightarrow: A \times A \to A$, such that $x \leq (y \Rightarrow z)$ iff $x \wedge y \leq z$ for all $x, y, z \in A$.*

In any Heyting algebra we can define a negation $\neg : A \to A$ by setting $\neg x$ to be $x \Rightarrow 0$. This negation will satisfy $x \wedge \neg x = 0$. In fact $\neg x$ is the largest element with this property in the sense that $x \wedge y = 0$ iff $y \leq \neg x$. However, the equation $x \vee \neg x = 1$ need not hold, nor need other equations, familiar in the Boolean case, such as $\neg\neg x = x$. Heyting algebras were originally developed by Heyting around 1930 to model the intuitionistic propositional calculus, and, in this context, $x \vee \neg x = 1$ is the law of the excluded middle, which is not valid in intuitionistic logic.

An example of a Heyting algebra is provided by the set of open sets of a topological space. In this case $\wedge$, $\vee$, 0, and 1 are interpreted as in the powerset case, but $\neg x$ is the interior of the complement of x, or equivalently, the complement of the closure of x. Such a Heyting algebra is always complete.

Definition 4. *A **co-Heyting algebra** is a distributive lattice equipped with a binary operation "subtraction", $\smallsetminus : A \times A \to A$ such that $x \smallsetminus y \leq z$ iff $x \leq y \vee z$.*

In any co-Heyting Algebra we can define an operation "supplement" $\sim : A \to A$, by setting $\sim x = 1 \smallsetminus x$. This operation will satisfy $x \vee \sim x = 1$, and $\sim x$ is the least element with this property in the sense that $x \vee y = 1$ iff $\sim x \leq y$. However, the dual equation $x \wedge \sim x = 0$ need not hold.

One example of a co-Heyting algebra is provided by the set of closed sets of a topological space. In this context $\sim x$ is the complement of the interior of x, making $\sim\sim x$ the closure of the interior of x. The boundary of a closed set x is given by $x \wedge \sim x$.

Any Boolean algebra is both a Heyting algebra and a co-Heyting algebra. We can define $x \Rightarrow y = \neg x \vee y$ and $x \smallsetminus y = x \wedge \neg y$. The two negations obtained from these defined operations i.e. $x \Rightarrow 0$ and $1 \smallsetminus x$ both coincide with the negation $\neg$ present in the Boolean algebra. In general, a lattice which is both a Heyting algebra and a co-Heyting algebra is called a **bi-Heyting algebra**. In an arbitrary bi-Heyting algebra the equations $x \Rightarrow y = \neg x \vee y$ and $x \smallsetminus y = x \wedge \sim y$ need not hold.

3 Bi-Heyting Algebras as Algebras of Regions

If we regard a set of regions as a bi-Heyting algebra, then we can express certain important constructions on the regions purely in terms of the operations present in the algebra. In this section we illustrate some of these constructions by giving two simple examples where the elements of a bi-Heyting algebra can be seen as spatial or temporal regions.

The examples in this section are bi-Heyting algebras by virtue of some general theory the details of which are not important for the rest of this paper. Specifically, in any presheaf topos the lattice of subobjects of any object forms a bi-Heyting algebra. For details of this theory see [RZ96]. The mathematical content of our examples is not original: closely related material can be found in [RZ96,Law91]. We present these examples here as evidence to support our contention that the algebraic structure of a bi-Heyting algebra is well suited to the manipulation of some sets of regions which occur in spatial information systems. This point does not seem to have been made before.

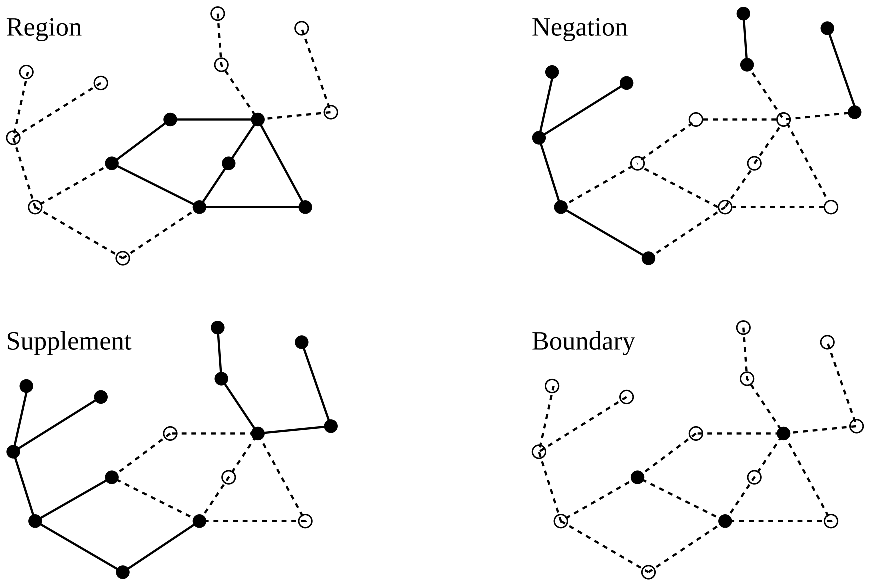
3.1 Example 1: Regions in Networks

A network can be modelled as an undirected graph $G = \langle G_N, G_A \rangle$ having a set, G_N, of nodes and a set, G_A, of arcs. Every arc $a \in G_A$ has a set of two end nodes, or just one in the case of a loop. A set, A, of arcs determines a set of nodes, viz. those nodes which are an end of some arc in A, we will denote this set by $\mathsf{ends}\, A$. Dually, a set, N, of nodes determines a set $\mathsf{arcs}\, N$, containing all arcs with both ends in N. A region is an arbitrary sub-graph of G, the empty graph is allowed as a region.

The set of regions is a bi-Heyting algebra. Given regions, $R_1 = \langle N_1, A_1 \rangle$, and $R_2 = \langle N_2, A_2 \rangle$, their meet is $R_1 \wedge R_2 = \langle N_1 \cap N_2, A_1 \cap A_2 \rangle$, and their join is $R_1 \vee R_2 = \langle N_1 \cup N_2, A_1 \cup A_2 \rangle$. Subtraction and implication can be defined, but the details are not important here. From the subtraction and implication, we obtain the negation, $\neg\langle N,\ A\rangle = \langle N',\ \mathsf{arcs}(N')\rangle$, and the supplement $\sim\langle N,\ A\rangle = \langle N' \cup \mathsf{ends}(A'),\ A'\rangle$ where N' is the set-theoretic complement of N in G_N, and analogously for A'.

Given a region, R, the expression $R \wedge {\sim} R$ denotes the nodes in R which are connected to nodes outside R. These nodes constitute the boundary, ∂R, of R. Note that if $\sim$ were the negation in a Boolean algebra, $R \wedge {\sim} R$ would be a contradiction, i.e. false. In the more general setting of a co-Heyting algebra, and interpreting the elements as regions, $R \wedge {\sim} R$ is a region lying both within R and outside R.

The following diagrams show a region together with its negation, supplement and boundary. In each diagram, the region in question is highlighted by using solid arcs and nodes.



Starting with a region R we can construct other regions using the operations defined above. A selection of useful constructions is given in the following table. This provides some convincing evidence of the expressiveness of the algebraic operations in this context.

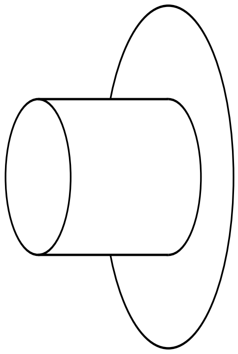
Described Region	Expression
R together with things accessible from R along one arc.	${\sim}\neg R$
Things inside R at least one arc away from ∂R.	$\neg{\sim} R$
Things at most one arc away from the boundary.	${\sim}\neg{\sim} R \wedge {\sim}\neg R$
R without any nodes which are not an end of an arc in G.	${\sim}{\sim} R$
R together with all arcs in G whose ends lie in R.	$\neg\neg R$

In the table, the expression $\sim\neg\sim R \wedge \sim\neg R$ can be seen as a 'thickened boundary' i.e. things not more than one arc away from the boundary. It is straightforward to generalize this to yield a sequence of progressively 'thicker' or 'broader' boundaries $\partial_1 R$, $\partial_2 R$, $\partial_3 R$ etc. where $\partial_n R$ contains everyting at most n arcs away from ∂R.

3.2 Example 2: Two-Stage Sets

Let A be a set and define L to be the set of pairs of subsets of A, $\langle x, y\rangle$ such that $x \subseteq y$. The set A could have some structure, perhaps as a geometrical region, but the discussion below does not assume this.

We can think of an element of L in a number of ways which are relevant to spatial and temporal reasoning. One view is that $\langle x, y\rangle$ is a set seen at two times, and that it has grown over some two stage process, x being the set at the start and y that at the end. Another interpretation involves regions with indeterminate boundaries. If $R = \langle x, y\rangle$ then x consists of those things which are certainly in R, whereas y consists of those things which maybe in R in some sense. We have found it helpful to visualize $\langle x, y\rangle$ as a flanged cylinder with discs for the sets x and y, and the solid part of the cylinder for the inclusion function $x \hookrightarrow y$.



The definitions of the operations $\vee$, and $\wedge$, and the elements 0 and 1 are straightforward. $\langle\, x_1,\ y_1\,\rangle \wedge \langle\, x_2,\ y_2\,\rangle = \langle\, x_1 \cap x_2,\ y_1 \cap y_2\,\rangle$, $\langle\, x_1,\ y_1\,\rangle \vee \langle\, x_2,\ y_2\,\rangle = \langle\, x_1 \cup x_2,\ y_1 \cup y_2\,\rangle$, $0 = \langle\, \varnothing,\ \varnothing\,\rangle$, and $1 = \langle\, A,\ A\,\rangle$. Subtraction is defined by taking $\langle\, x_1,\ y_1\,\rangle \smallsetminus \langle\, x_2,\ y_2\,\rangle$ to be $\langle\, x_1 \cap x_2',\ (x_1 \cap x_2') \cup (y_1 \cap y_2')\,\rangle$, where the prime, $'$, denotes the complement in A. Implication is defined by $\langle\, x_1,\ y_1\,\rangle \Rightarrow \langle\, x_2,\ y_2\,\rangle = \langle\, (x_1' \cup x_2) \cap (y_1' \cup y_2),\ y_1' \cup y_2\,\rangle$. These give rise to the supplement $\sim\langle\, x,\ y\,\rangle = \langle\, x',\ x'\,\rangle$, and the negation $\neg\langle\, x,\ y\,\rangle = \langle\, y',\ y'\,\rangle$.

As with the network example, the algebraic operations provide an expressive language in which many intuitively meaningful constructions can be formalized. Two of particular significance are the core and the boundary, which can easily be visualized in connection with the flanged cylinder picture.

CORE	$\sim\sim\langle\, x,\ y\,\rangle = \langle\, x,\ x\,\rangle$	BOUNDARY	$\langle\, x,\ y\,\rangle \wedge \sim\langle\, x,\ y\,\rangle = \langle\, \varnothing,\ x' \cap y\,\rangle$

If we think of $\langle\, x,\ y\,\rangle$ as a set growing over a two stage process, then the core corresponds to the elements which are present at the start, and remain present during the process. The boundary corresponds to the elements which are added during the process, i.e. those present at the end but not the start. If we interpret $\langle\, x,\ y\,\rangle$ as a region where there is some doubt as to which things are in the region, then the core corresponds to the elements definitely present, whereas the boundary is those things which both might and might not be included.

3.3 Interpretations of Algebraic Constructions

Having discussed just two examples, it may be helpful to draw the reader's attention to a potential misconception. It can be misleading to interpret the same construction in different bi-Heyting algebras as having the same intuitive meaning. In both the two examples we have discussed, $R \wedge \sim R$ does correspond to a meaningful boundary. However, in the two stage sets the boundary is disjoint from the core: $\partial R \wedge \sim\sim R = 0$, but in the network example this is rarely the case.

4 Modelling the Region-Connection Calculus without using Points

There have been several investigations [GGC96,FE91,FE95,Smi96] devoted to the question of how to characterize the essential properties of regions as used in spatial information systems. It has been observed that classical mathematical models of space, in particular point-set topology, are unsatisfactory for many practical aspects of spatial reasoning. This issue is discussed in detail in [GGC96].

4.1 The Region-Connection Calculus

The Region-Connection Calculus (RCC) has been developed by Cohn and his colleagues [GGC96], based on the work of Clarke [Cla81]. RCC is a formalism which treats regions as primitive, instead of as sets of more basic points. The theory uses a binary relation, C, of "connection" which may hold between regions, partial binary operations sum and prod, and a unary operation compl. Eight axioms are expressed in a first-order sorted logic using just the connection relation and the three operations. The axioms have intuitive meanings in which points play no role.

In a recent report [Got96], Gotts considers the question of models for the RCC axioms. Gotts shows that certain topological spaces, the T_3 connected ones, provide models of the RCC axioms by taking a region to mean a non-empty regular closed set. Taking non-empty regular open sets also gives a model. These models have the disadvantage that in justifying them, reference is made to points within regions. Gotts observes that *"Using an interpretation expressed in terms of point-sets might seem inconsistent with the spirit underlying the RCC approach. However no alternative has been worked out in any detail, ... "*. In this section we provide a treatment of a class of models for the RCC axioms which does not suffer from this disadvantage. Our models include the ones provided by Gotts, but the important feature of our work is that we can prove that our regions model the axioms without mentioning points in regions.

4.2 Pointless Topology

As already noted in section 2 above, the open sets of a topological space form a complete Heyting algebra. One approach to topology is obtained by simply defining a space to be a complete Heyting algebra. We can then study a form of topology by using just the lattice theoretic properties of the collection of open sets, without mentioning points at all. This approach is often called "pointless topology", but the reader should note that this does not mean that points are prohibited, only that points are not required. Pointless topology is more general than point-set topology in that there are complete Heyting algebras which do not come from the open set lattices of topological spaces. On the other hand, there are aspects of topological spaces which depend critically on points, and which cannot be described in terms of the open sets alone. Pointless topology has been well developed by mathematicians [Joh82,Vic89], but the subject does not seem to have received much attention in the spatial reasoning community.

4.3 A Class of Models of the RCC Axioms

In a Heyting algebra, A, we say a **is well inside** b iff $\neg a \vee b = 1$. A complete Heyting algebra is defined to be **regular** if every element $a \in A$ is the join of elements well inside it:

$$a = \bigvee \{x \in A \mid \neg x \vee a = 1\}.$$

A Heyting algebra is said to be **connected** if it does not contain elements $a \neq 0$ and $b \neq 0$ such that $a \vee b = 1$ and $a \wedge b = 0$. Note that any Boolean algebra is regular but cannot be connected if it has more than two elements.

Now let A be a regular, connected, complete Heyting algebra containing more than two elements, and define $R = \{a \in A \mid \neg\neg a = a \text{ and } a \neq 0\}$. We will show that R provides a model of the RCC axioms. The lattice of open sets of any connected T_3 topological space, having more than two open sets, is a Heyting algebra with the required properties. Thus our result includes that of Gotts.

The relation $\mathsf{C}(x, y)$ is defined to be $\neg x \vee \neg y \neq 1$. The three operations required are defined by $\mathsf{sum}(x, y) = \neg\neg(x \vee y)$, $\mathsf{prod}(x, y) = x \wedge y$, and $\mathsf{compl}(x) = \neg x$. These operations do yield elements which satisfy $\neg\neg a = a$. Checking this requires the facts that in any Heyting algebra $\neg\neg\neg a = \neg a$, and $\neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b$. However, the equation $\neg\neg(a \vee b) = \neg\neg a \vee \neg\neg b$ fails in general, which explains why $\mathsf{sum}(x, y)$ has to be defined as $\neg\neg(x \vee y)$ and not as $x \vee y$.

In RCC various relations are defined in terms of C. These include the following, which are intended to model the ideas of part (P), proper part (PP), overlap (O), external connection (EC), and non-tangential proper part (NTPP).

$\mathsf{P}(x, y)$	iff	For every region z, $\mathsf{C}(z, x)$ implies $\mathsf{C}(z, y)$
$\mathsf{PP}(x, y)$	iff	$\mathsf{P}(x, y)$ and not $\mathsf{P}(y, x)$
$\mathsf{O}(x, y)$	iff	There is some region z such that $\mathsf{P}(z, x)$ and $\mathsf{P}(z, y)$
$\mathsf{EC}(x, y)$	iff	$\mathsf{C}(x, y)$ and not $\mathsf{O}(x, y)$
$\mathsf{NTPP}(x, y)$	iff	$\mathsf{PP}(x, y)$ and there is no z such that $\mathsf{EC}(z, x)$ and $\mathsf{EC}(z, y)$

The bulk of the work in verifying the RCC axioms is in establishing the following description of these relations in terms of the operations of the Heyting algebra.

Theorem *Let A be any complete, regular, connected Heyting Algebra, and define the set of regions by $R = \{a \in A \mid \neg\neg a = a \text{ and } a \neq 0\}$. Define the relation C on R by $\mathsf{C}(x,y)$ iff $\neg x \vee \neg y \neq 1$. Then the defined relations can be characterized as follows.*

$$\begin{array}{ll} \mathsf{P}(x,y) & \textit{iff } x \leq y \\ \mathsf{PP}(x,y) & \textit{iff } x < y \\ \mathsf{O}(x,y) & \textit{iff } x \wedge y > 0 \\ \mathsf{EC}(x,y) & \textit{iff } \neg x \vee \neg y < 1 \textit{ and } x \wedge y = 0 \\ \mathsf{NTPP}(x,y) & \textit{iff } \neg x \vee y = 1 \textit{ and } x \neq y \end{array}$$

Proof We restrict ourselves to the characterizations of P and NTPP. The other parts are much more straightforward.

To prove that $\mathsf{P}(x,y)$ iff $x \leq y$, suppose first that $x \leq y$. Then $\neg y \leq \neg x$, so, for any z, $\neg z \vee \neg y \leq \neg z \vee \neg x$. Now, if $\neg z \vee \neg y = 1$, we must have $\neg z \vee \neg x = 1$. Thus, for any z, not $\mathsf{C}(z,y)$ implies not $\mathsf{C}(z,x)$, so $\mathsf{P}(x,y)$ holds.

Conversely, suppose that $x \not\leq y$. Consider the set $\{w \in A \mid \neg w \vee \neg y = 1\}$ of elements well inside $\neg y$. If everything well inside $\neg y$ is also well inside $\neg x$, then $\{w \in A \mid \neg w \vee \neg y = 1\} \subseteq \{w \in A \mid \neg w \vee \neg x = 1\}$. But, as A is regular, taking the join of each of these sets gives $\neg y \leq \neg x$ which implies $\neg\neg x \leq \neg\neg y$ so $x \leq y$, as x and y are regions. Thus we can find w such that $\neg w \vee \neg y = 1$, but $\neg w \vee \neg x \neq 1$. This w must be non-zero, but it may fail to be a region as it may not satisfy $\neg\neg w = w$. Thus we define $z = \neg\neg w$, so that $\neg\neg z = z$. Now, z must be a region, for if $z = 0$, then $\neg\neg\neg w = \neg 0$, so $\neg w = 1$. But this implies $w \vee \neg w = 1$, which is impossible since A is connected. As $\neg z = \neg w$, we have $\neg z \vee \neg y = 1$, but $\neg z \vee \neg x \neq 1$. Hence we have a region z such that $\mathsf{C}(z,x)$ but not $\mathsf{C}(z,y)$, thus making $\mathsf{P}(x,y)$ false.

To prove that $\mathsf{NTPP}(x,y)$ iff $\neg x \vee y = 1$ and $x \neq y$, note first that $\mathsf{NTPP}(x,y)$ is equivalent, by virtue of the characterizations of PP and EC, to $x < y$ and there being no z with all the properties

$$\neg z \vee \neg x < 1, \quad z \wedge x = 0, \quad \neg z \vee \neg y < 1, \quad \text{and } z \wedge y = 0. \tag{1}$$

From $\neg x \vee y = 1$ and $x \neq y$ we get $\neg\neg x \wedge \neg y = 0$ so $x \wedge \neg y = 0$. Now $\neg x$ is the largest element meeting x in 0, so $\neg y \leq \neg x$, and so $x \leq y$. Hence we have $x < y$ since $x \neq y$. If there is z with the properties in (1), then from $z \wedge y = 0$, we get $y \leq \neg z$. But then $1 = \neg x \vee y \leq \neg x \vee \neg z$, and thus $\neg x \vee \neg z = 1$. Hence we conclude that no z with the stated properties exists.

Conversely, suppose that $\mathsf{NTPP}(x,y)$. From $x < y$ we get $x \neq y$ immediately, and we deduce that $\neg y \leq \neg x$. We need to obtain $\neg x \vee y = 1$, so suppose to the contrary that $\neg x \vee y < 1$. We can then find z satisfying (1) by taking $z = \neg y$. This contradicts $\mathsf{NTPP}(x,y)$, so we conclude that $\neg x \vee y < 1$ and $x \neq y$. ∎

Having proved the theorem, it remains to check each of the eight RCC axioms. This uses arguments similar to those presented above, and is mostly straightforward. Further details are omitted, and a fuller account of this work, in a more general context, can be found in Stell's report [Ste97].

5 Regions with Indeterminate Boundaries

The intuition behind the approach discussed in this section, is that the certainty of membership of a location in a region is determined by observing the relationship of the region to neighbourhoods of the location. We call these neighbourhoods environments. To take a simple case, suppose that an environment of a particular location is completely contained in the region, then we can say that the location is certainly in the region. However, if the environment overlaps the region, we can say that the location is possibly in the region, and if the environment is disjoint from the region, then the location is certainly not in the region. In general, a given location will have several environments, whose relationships with the region provide evidence for or against the location being in the region. The algebraic framework that we develop below formalizes these notions. This approach, unlike fuzzy logic, has the advantage that it retains a fully qualitative approach to membership conditions for regions with indeterminate boundaries. The idea is that although we may not be able to directly determine with certainty whether a location is or is not in a region, we may be able to say something about the relationships of environments of the location to the region, and this may provide indirect evidence about the membership of the location in the region. For example, we can say with a high degree of certainty that Nelson's Column is in 'the south of England' (a region with indeterminate boundary), since there are many environments of Nelson's Column, for example London, that are deemed to be wholly in 'the south of England'. However, Oxford being in 'the south of England' is less clear, as there are many environments of Oxford, for example the Oxford, Newbury, Reading triangle, that might be thought to overlap but not be wholly contained within 'the south of England'. Edinburgh will certainly be deemed not to be in 'the south of England', since many of its environments are not even in England, and so disjoint from 'the south of England'. Of course, it all depends on what are taken as the environments.

We now proceed to formalize some of these ideas, linking them to algebraic structures discussed earlier in the paper. Let the set X be given, along with a family $\mathcal{E}$ of non-empty subsets of X, called environments, that covers X in the sense that every member of X is in at least one environment. For $x \in X$, if x is a member of environment E, we say that E is an environment of x. Let R be a region with indeterminate boundary (RIB), we introduce two modal operators that return regions with points that have a high degree of certainty to be in R.

$$\begin{aligned} \Box R &= \{x \in X \mid \text{ there is an environment } E \text{ of } x \text{ such that } E \subseteq R\} \\ \blacksquare R &= \{x \in X \mid \text{ all environments } E \text{ of } x \text{ are such that } E \subseteq R\} \end{aligned}$$

Note that $\blacksquare R \subseteq \Box R \subseteq R$, and that if $\mathcal{E}$ is a partition of X, then $\blacksquare R = \Box R$. In general, if $\mathcal{E}$ is a rich set of environments, with many environments of each location, $x \in \blacksquare R$ is an extremely restrictive condition. Dually, we can define two further modal operations that return regions whose points may possibly belong to R.

$$\begin{aligned} \blacklozenge R &= \{x \in X \mid \text{ there is an environment } E \text{ of } x \text{ such that } E \cap R \neq \varnothing\} \\ \Diamond R &= \{x \in X \mid \text{ all environments } E \text{ of } x \text{ are such that } E \cap R \neq \varnothing\} \end{aligned}$$

Note that $R \subseteq \Diamond R \subseteq \blacklozenge R$, and that if $\mathcal{E}$ is a partition of X, then $\Diamond R = \blacklozenge R$. Our definitions of modal operators, $\Box$ and $\Diamond$, in terms of $\mathcal{E}$, substantially generalize those found in [MR94, page 86], where $\mathcal{E}$ consists of the squares of a grid covering the plane.

There are relationships between the box and diamond operators as follows, where R' indicates the complement of the set R.

$$(\Box R')' = \Diamond R, \qquad (\Diamond R')' = \Box\Box R, \qquad (\blacksquare R')' = \blacklozenge R, \qquad (\blacklozenge R')' = \blacksquare\blacksquare R$$

It is not difficult to show that $\Box\Box R = \Box R$ and $\Diamond\Diamond R = \Diamond R$. Our next step is to introduce two forms of negation, and relate them to the modal operators.

$$\begin{aligned} \neg R &= \{x \in X \mid \text{ there is an environment } E \text{ of } x \text{ such that } E \cap R = \varnothing\} \\ \sim R &= \{x \in X \mid \text{ all environments } E \text{ of } x \text{ are such that } E \nsubseteq R\ \} \end{aligned}$$

Note that $\neg R = \Box R'$, and $\sim R = \Diamond R'$, so we have $\neg\!\sim R = \Box\Box R$ and $\sim\neg R = \Diamond\Diamond R$.

Since $\Box\Box R = \Box R$ and $\Diamond\Diamond R = \Diamond R$ we have that $\Box R = \neg\!\sim R$ and $\Diamond R = \sim\neg R$. The general idea that modal operators can be related to $\sim$ and $\neg$ in this way, was first explored by Reyes and Zolfaghari [RZ96].

6 Conclusions and Further Work

This paper has demonstrated that concerns of central importance to spatial information theory can profitably be addressed in terms of Heyting algebras and related structures. Of particular significance is the fact that rigorous models of regions satisfying the RCC axioms can given in a way which completely avoids the use of points within regions. Further work on models of the RCC axioms from the perspective of pointless topology should provide valuable insights into what form other models of the axioms may take.

In the qualitative theory of regions with indeterminate boundaries initiated in section 5, implication and subtraction operations may be defined as follows. For RIBs R and S, define $R \Rightarrow S$ to be $\{x \mid$ there is an environment E of x such that $E \subseteq R' \subseteq S\}$, and $R \smallsetminus S$ to be $\{x \mid$ for all environments E of x, $E \nsubseteq R' \subseteq S\}$. The set of RIBs need not be a Heyting or a co-Heyting algebra, but we believe that this will be the case when extra conditions are imposed on the sets of environments. Future work will investigate the algebraic structure of the set of RIBs, and will consider the properties of the modal operators. The work will also consider applications of these results for reasoning about RIBs.

Other further work on regions with indeterminate boundaries is suggested by the two examples in section 3. The thickened boundaries in the network example appear to have connections with the "broad boundaries" of Clementini and di Felici [CdF96]. The two-stage sets can also be understood as regions with indeterminate boundaries. We intend to explore this interpretation in the light of Cohn and Gotts' work [CG96] on the "egg–yolk" approach and its relationship to the RCC axioms.

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