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The Poisson Process as a Model for a Diversity of Behavioral Phenomena.

By

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1. The Poisson process - Telephone calls.

In order to illustrate the basic principles of the Poisson process I may refer to one of its most popular applications, viz. the theory for incoming calls to a telephone exchange.

Whenever you call somebody on the phone, you may have a very good reason for doing so, you may feel that you can really give a valid causal explanation of your call. But for the telephone exchange your causality is of no import whatsoever. Your call is just one among a thousand other incoming calls which to all intents and purposes of the telephone exchange may be described as a series of independent random events, occurring with a constant intensity over a stable period.

This point of view may be formalized as follows:

Within the period considered the probability of an incoming call is λdt for any differential time interval $(t, t + dt)$, the intensity λ being constant throughout the whole period; in particular the probability is unaffected by the number of calls that preceded t and by the time elapsed since the last call as well. In addition we assume, that the probability of two calls in the same differential interval is infinitely small as compared with λdt.[1]

From this set of assumptions it follows that the probability of a calls in a finite period T is given by the Poisson distribution with the mean value

$$\mu = \lambda T, \tag{1.1}$$

i.e.

$$p\{a|\lambda, T\} = e^{-\lambda T} \cdot \frac{(\lambda T)^a}{a!}. \tag{1.2}$$

1) See e.g. Feller [5], p.400-402. 2) See e.g. Rasch [10], p.129-130.

On this occasion I do not intend to present a statistical documentation of how well the Poisson process works in this field - you may easily dig up any amount of data - but in view of what follows I may mention a couple of properties of the Poisson distribution upon which such a control may be based, viz. <u>the rules of additivity and conditionality:</u>

If a_1 and a_2 are independent random variables following Poisson distributions with the parameters μ_1 and μ_2, then the sum

(1.3) $$a_o = a_1 + a_2$$

also follows Poisson distribution, the mean value of which is

(1.4) $$\mu_o = \mu_1 + \mu_2,$$

and furthermore the conditional distribution of, say, a_1 in such pairs with given sum (1.3) will follow a Binomial distribution:

(1.5) $$p\{a_1 | a_o, \mu_1, \mu_2\} = \binom{a_o}{a_1} \left(\frac{\mu_1}{\mu_o}\right)^{a_1} \left(\frac{\mu_2}{\mu_o}\right)^{a_2} .$$

2. Accident proneness. Shunters.

For incoming calls to telephone exchanges the Poisson process would seem a quite reasonable model, and its applicability has, in fact, been born out in practice.

In 1919 Greenwood and Woods [5] 1) suggested the same model as the basis for studying the accidents of 648 female workers in an English ammunition factory. From an examination of the distribution of the workers according to their number of accidents during 13 months the authors concluded that the intensity of accidents, λ, could not possibly have been common to all of the workers. There had to be a certain individual variation in λ which was then satisfactorily described. Thus the concept of "accident proneness" was brought forth.

[illegible]
1) ~~The paper is inaccessible to me, but is frequently~~ quoted.

However, to state that a number of persons do not have an intensity in common is very far from showing that any of them does possess such a parameter. In order to do so the authors should have established the Poisson process as a possible description of the series of accidents for each individual. This would require repeated observations on each worker and such were not available.
some were.

For 122 South African shunters Kerrich and Arbous reported the number of accidents in two periods of respectively $T_1 = 6$ and $T_2 = 5$ years.

Table 1.

A consequence of the hypothesis that shunter no. ν keeps his own "pronity for accidents", λ_ν, constant over $T_0 = 11$ years is that it applies to each of the periods, in which, then, the probability of his actual numbers of accidents, $a_{\nu 1}$ and $a_{\nu 2}$, should be given by the Poisson distribution with the mean values

(2.1) $$\mu_{\nu 1} = \lambda_\nu T_1 \text{ and } \mu_{\nu 2} = \lambda_\nu T_2.$$

His pronity could, according to the additivity rule, be estimated from the Poisson distribution for the total

(2.2) $$a_{\nu 0} = a_{\nu 1} + a_{\nu 2},$$

the mean value of which is

(2.3) $$\mu_{\nu 0} = \mu_{\nu 1} + \mu_{\nu 2} = \lambda_\nu T_0$$

so that

(2.4) $$\lambda_\nu = \frac{a_{\nu 0}}{T_0},$$

i.e. λ_ν is estimated by his number of accidents per time unit.

But does he keep a constant λ_ν - in particular, is it the same in the two intervals considered?

If so, it would follow from the conditionality rule that the probability of just $a_{\nu 1}$ accidents, out of his total of $a_{\nu 0}$, falling in the first period were given by the Binomial distribution with the parameter

$$(2.5)\qquad \frac{\mu_{\nu 1}}{\mu_{\nu 0}} = \frac{T_1}{T_0} = \frac{6}{11} .$$

This result is remarkable for two reasons. Firstly, the probability is independent of the particular shunters parameter λ_ν, i.e. it is common to all shunters which happened to have the same total number of accidents, e.g. for the 19 shunters with $a_{\nu 0} = 3$. Secondly, the parameter of the Binomial distribution is even known and the same for all values of $a_{\nu 0}$, viz. $= \frac{6}{11}$. Thus we may in the case of $a_{\nu 0} = 3$ compute the probabilities for $a_{\nu 1} = 0$, 1, 2 and 3 as shown in table 2.

Table 2.

On multiplying by 19 we get a calculated distribution showing a good fit with the observed one as seen from table 3.

Table 3.

For values of $a_{\nu 0}$ with few persons this procedure is ineffective, but we may in any case calculate a mean and a mean square. These figures are shown in table 4 which also gives a conversion of table 1.

Table 4.

As a consequence of (1.3) together with (2.5)

$$(2.6)\qquad \mathcal{M}\{a_{\nu 1} | a_{\nu 0}\} = a_{\nu 0} \cdot \frac{6}{11} = 0.545\, a_{\nu 0}$$

and

$$(2.7)\qquad \mathcal{V}\{a_{\nu 1} | a_{\nu 0}\} = a_{\nu 0} \cdot \frac{6}{11} \cdot \frac{5}{11} = 0.248\, a_{\nu 0} .$$

Thus, from this consequence of the model, it follows that by plotting the means and the mean squares against $a_{\nu 0}$ we should get points clustering around proportionality lines with the slopes o.545 and o.248. From fig. 1 a and 1 b it is clear that the observations pass this test excellently.

3. Accident proneness Busdrivers.

Farmer and Chambers [3][1] have provided such details about 5 years traffic accidents of 166 London busdrivers that the total number of accidents for each driver can be compared with his number of accidents in both the first year and the last year. These data have been presented as table 1o and 11 in a paper by Bates and Neymann [2] p. 242-244, and in the tables 5 and 6 below these tables have been recast into a form similar to table 4, showing the distribution of numbers of accidents in the first year, respectively the last year, for given totals in the 5 years.

Table 5 a.

In analogy to the precedinganalysis we have in fig. 2 a plotted the average of $a_{\nu 1}$ for given total $a_{\nu o}$ against the theoretical mean value

$$\mathfrak{M}\{a_{\nu 1} | a_{\nu o}\} = \frac{T_1}{T_o} a_{\nu o} = \frac{1}{5} a_{\nu o} \quad (3.1)$$

(cf. (2.6)).

As a conspicuous number of points lie above the identity line, a more exact test is needed. From the theory of the Binomial distribution it is known that the grand total of the $a_{\nu 1}$'s should concord with the distribution

$$p\{a_{o1} | a_{oo}\} = \binom{a_{oo}}{a_{o1}} \theta^{a_{o1}} (1-\theta)^{a_{oo}-a_{o1}} \quad (3.2)$$

where a_{oo} is the grandtotal of the $a_{\nu o}$'s and where $\theta = \frac{1}{5}$. Thus a_{o1} = 3o1 should estimate $\frac{1}{5} a_{oo} = \frac{1}{5} \cdot 133o = 266.o$, but the difference ~~of~~ from 35o amounts to 2.4 x the standard error which is $\sqrt{133o \cdot \frac{4}{5}}$. As a consequence we have to reject ~~aver~~ they basic hypo-

1) The paper is inaccessible to me, but is frequently quoted.

thesis of a constant (personal) pronity for each individual.

For a modification the data themselves are suggestive. In fact, structurally the observed distributions of $a_{\nu 1}$ for given $a_{\nu 0}$, as shown in table 5 a, would seem to agree with Binomial distributions with a parameter in common, which then would have to be estimated from the data.

If accepted, this suggestion implies, of course, that part of our model breaks down. But no more is required for its restoration than replacing the constant intensity λ_ν of (2.1) by one pronity, $\lambda_{\nu 1}$, for the first period and another one, $\lambda_{\nu 2}$, for the remaining period, the two pronities having a constant ratio. Writing, accordingly,

(3.3) $$\lambda_{\nu i} = \xi_\nu \varepsilon_i \text{ and } \mu_{\nu i} = \xi_\nu \varepsilon_i \cdot T_i = \xi_\nu \tau_i ,$$

it follows from the rule of conditionality that the distribution of $a_{\nu 1}$ for given total $a_{\nu 0}$ is binomial with the parameter

(3.4) $$\frac{\mu_{\nu 1}}{\mu_{\nu 0}} = \frac{\tau_1}{\tau_0} = \frac{\varepsilon_1 T_1}{\varepsilon_1 T_1 + \varepsilon_2 T_2} .$$

In the actual comparison of the first year to the total this parameter is estimated at

(3.5) $$\frac{\tau_1}{\tau_0} \approx \frac{301}{1330} = 0.226.$$

Table 6 enables us to compare the fifth year to the total period, giving

(3.6) $$\frac{\tau_5}{\tau_0} \approx \frac{255}{1330} = 0.192.$$

Denoting further the τ for the intermediate 3 years by τ_2 we get

(3.7) $$\frac{\tau_2}{\tau_0} \approx \frac{1330-556}{1330} = \frac{774}{1330} = 0.582$$

and on dividing

(3.8) $$\frac{\tau_2}{\tau_5} \approx \frac{774}{255} .$$

Now

$$\frac{r_2}{r_5} = \frac{\varepsilon_2 T_2}{\varepsilon_5 T_5} = 3\,\frac{\varepsilon_2}{\varepsilon_5}. \tag{3.9}$$

Accordingly

$$\frac{\varepsilon_2}{\varepsilon_5} \approx \frac{774}{3 \cdot 255} = 1.o12 \tag{3.1o}$$

which means that the pronity in the years 2-4 is practically identical to the pronity in the fifth year.

Thus we are left with a contrast between the first year and the four succeding years, with a pronity ratio of

$$\frac{\varepsilon_1}{\varepsilon_{2-5}} \approx 4 \cdot \frac{3o1}{1o29} = 1.17, \tag{3.11}$$

i.e. according to the model the pronity should for all busdrivers concerned have been some 17 o/o higher in the first year of observation than later on.

Before accepting this conslusion we must, however, consider the statistical evidence regarding the model as such.

In table 5 b we have calculated the mean square (MS) of the $a_{\nu 1}$'s for each $a_{\nu o}$, to be compared with the binomial variance

$$\sigma^2 = a_{\nu o} \cdot \theta(1-\theta) \tag{3.12}$$

as estimated at

$$s^2 = o.175\, a_{\nu o} \tag{3.13}$$

by substituting o.226 for θ. From fig. 2 b it is seen that MS does not deviate systematically from s^2, and by pooling, as indicated in the last column of the table, we get the estimate 26.16/145=o.18o which compares favourably with o.226 · o.774 = o.175. A similar result is obtained from table 6 as regards the last year. The factor is estimated at 22.76/145 = o.157, to be compared with o.192 · o.8o8 = o.155.

Thus we have demonstrated that for given $a_{\nu o}$ the variation of both $a_{\nu 1}$ and $a_{\nu 5}$ is just what it should be like according to the model.

Statistically, therefore, the conclusion reached at would seem well founded, in so far as the modified model is at all acceptable.

Psychologically, however, it would seem almost unbelievable that all of the drivers should all of a sudden reduce their respective pronities by the same relative amount. But the same numerical effect might be produced through a reduction of the working hours - for instance from 9 to 8 hours a day - which would deminish the effective exposition time during the last 4 years. However, I don't know if such an event did happen by that time. Even if it didn't the model might be good enough for describing a certain training effect, more or less common to all of the drivers, in which case the ratio a_{2-5}/a_1 could be perceived as some sort of average effect.

It is quite conceivable that a psychologically more sound model could be advanced, but since the data showed such exellent fit to the present model no material gain could be expected from a more complex model, requiring more parameters unless more detailed observations became available.

4. Two behavioral experiments. Time observations.

An obvious way of refining the counting of occurrences would be to record the exact time for each occurrence. Such observations were made in two series of behavioral experiments carried out by the Danish psychologists Gerhard Nielsen and Iven Reventlow. In one series the subjects were students at Harvard university [6] 1), in the other series male sticklebacks [10].

In the human experiments each student was involved in a hot discussion on his philosophy of life and his behavior during this session was taken by sound film. A few days later the scene was played back to him and his behavioral responses to this self-confrontation were also recorded. In particular the time intervals in which, during the latter performance, he looked at himself, as well as the intermediate intervals in which he looked away from himself were recorded.

1) pp. 34-36, 48-49.

In the fish experiment each individual was swimming in a basin containing its nest, and under various conditions the time points for its swimming to the nest and for its leaving the nest were noted.

In both instances the behavior of a subject vacillates between two possible states - looking at self and looking away from self, staying at the nest and staying away from the nest.

By way of an example the sequence of time intervals in which a student (Hiber)[1] looked at and looked away from himself is given in table 7, and in fig. 3 these intervals are plotted against the observation number. As the two sequences of points, plotted upwards and downwards, look rather irregular - and, as a matter of fact, we should find the same sort of picture in case of the sticklebacks - we are going to conceive the changes in behavior as a sequence of random events.

When attempting to give a description of such sequences we shall set out from the same model as before, the Poisson process with constant intensity, but as the observations are now time intervals we have to shift over from numbers to durations as random variables.

This is fairly easily done. In fact, the Poisson Law states for a = o that the probability of no occurrence during the interval (o,T) is $e^{-\lambda T}$, which, then, is also the probability that the duration t of an occurrence-free period exceeds T, i.e.

$$P\{t \geq T\} = e^{-\lambda T} \quad . \tag{4.1}$$

Accordingly

$$\log(1/P\{t \geq T\}) = \lambda T \quad . \tag{4.2}$$

and furthermore

$$\log\log(1/P\{t \geq T\}) = \log\lambda + \log T \quad . \tag{4.3}$$

Now from table 7 we may compile two distributions, one of periods in which the testee looks at himself, one of the periods when he is looking away from himself. The corresponding cumulated distributions, counted from the top, are also shown in table 8 a, b.

1) Gerhard Nielsen [7].

Clearly, the relative frequency of observed t's exceeding any given T, $H_N\{t \geq T\}$, N being the number of observations, will estimate the corresponding probability (4.1), i.e.

$$(4.1a) \qquad H_N\{t \geq T\} \approx e^{-\lambda T}$$

and consequently

$$(4.3a) \qquad \log\log(1/H_N\{t \geq T\}) \approx \log\lambda + \log T,$$

in so far as the Poisson process holds.

Accordingly the tables also include the $\log\log(1/H_N)$-values, to be plotted in fig. 4 against $\log(t-1/2)$[x]. It is seen that we actually get points which - in agreement with (4.3a) - fairly closely fit a straight line with unit slope.

The same sort of result is obtained from a distribution from the stickleback experiments as demonstrated in table 9 and fig. 5.

Consider, however, another couple of selected cases, one from each experimental series, cf. the tables 10 and 11 together with figs. 6 and 7. Again the points cluster around straight lines, but now the slope for the stickleback unquestionably exceeds 1 and for the student it is clearly less than 1. And those two cases are in fact typical for the two sets of data:

For the sticklebacks we quite often find a slope of 1, but equally often it exceeds 1, and only in a few exceptional instances the slope is < 1.

For the students unit slope is also often found, but more often it differs from 1 and then it is regularly < 1, with only 3 or 4 exceptions out of 80 curves.

Thus we have reached at an empirical modification of (4.3a), namely

$$(4.4) \qquad \log\log(1/H_N\{t \geq T\}) \approx \log\lambda - \alpha\log T$$

where α is some positive parameter which may vary both within and between individuals.

x) t, as measured in sec.'s, is taken to cover the interval $(t-\frac{1}{2}, \; t+\frac{1}{2})$.

5. The Poisson process with time-dependent intensity.

Going back from (4.4) to anti-anti-logarithms and formalizing frequencies to probabilities we get as a substitute for (4.1)

$$P\{t \geq T\} = e^{-\lambda T^{\alpha}} . \tag{5.1}$$

In a certain sense this cumulated distribution function may be held to be of the same type as (4.1), of which it is a generalization. The only, but distinctive difference is that the time t is now measured inequidistantly through the transformation into a power t^{α}.

On differentiating (5.1) we get the distribution function (or probability density) of the duration t of a state:

$$p\{t\} = \lambda\alpha t^{\alpha-1}e^{-\lambda t^{\alpha}} . \tag{5.2}$$

Fig. 8 illustrates the 3 possibilities:

When $\alpha<1$ short durations are relatively frequent - the upper branch of the curve receding to ∞ as t approaches 0 - but large durations may very well occur.

For $\alpha = 1$ $p\{t\}$ tends to a finite value (λ) when $t \longrightarrow 0$ and it decreases monotonically - in fact exponentially - with increasing t.

Finally with $\alpha>1$ $p\{t\}$ tends to 0 as $t \longrightarrow 0$, so that small durations are relatively infrequent; a maximum is attained somewhere - depending in a complex way on λ and α - and larger values of t are, of course, getting rare.

So much for a description of the distributions.

Now divide $P_+\{t\}$, i.e. (5.1) with T replaced by t, into $p\{t\}dt$ to obtain the conditional probability that the state changes between t and t+dt, provided it has already lasted until t.

This leads to the general definition of the intensity function.

$$\lambda'(t) = \frac{p\{t\}}{P_+\{t\}} = -\frac{d\log P_+\{t\}}{dt} \tag{5.3}$$

which in the Poisson process was constant ($=\lambda$), but in the case now at issue it is proportional to a power of the time already spent by the individual in the state considered:

$$\lambda'(t) = \lambda\alpha t^{\alpha-1} . \tag{5.4}$$

The three possibilities $\alpha \lesseqgtr 1$ are illustrated in fig. 9.

The implications of this type of dependence are that for $\alpha > 1$ - as often happens in case of the behavior of the sticklebacks - the probability of a change in the next moment increases with the duration of the state. For $\alpha < 1$ - as is often found in the self-confrontation of the Harvard students - the probability of a change is large to begin with, but it decreases the longer the state has been endured. In the intermediate case, $\alpha = 1$, which is frequent in both investigations, the probability stays constant, irrespective of how long or how short the state has already lasted.

6. Criminal careers.

The technique developed in sect. 4 has tentatively been applied to the study of criminal careers collected under the direction of K.O. Christiansen [3] of Criminalistic Institute, Copenhagen.

Usually such careers are based upon the dates of the sentences and the periods of internment. However, for a special study on the criminality of twins also the dates of the offenses on record were ascertained.

In table 12 and fig. 9 a case with an exeptionally large number of sentences and offenses is analyzed with respect to the model suggested. The data are the net periods (in months) between successive sentences and successive offenses, respectively, the gross periods in both cases being reduced for intervening internments in order to obtain the periods of exposition.

In both records the points cluster rather closely around straight lines, and the same result has been found in some 40 careers that were substantial enough for an analysis of the distributions.

Accordingly it seems worth while to take the model (5.1) as a basis for the analysis of individual criminal careers, also in the majority of cases where there is only a very limited number of sentences. In these cases, of course, the graphical control on the model breaks down, but a numerical estimation of the parameters may be based on the formulae for the mean value (or expectation) and the variance of logt:

(6.1) $$\mathfrak{M}\{\log t\} = \frac{\gamma + \log \lambda}{\alpha}$$

and

(6.2) $$\mathcal{V}\{\log t\} = \frac{\pi^2}{6\alpha^2}$$

γ denotes Euler's constant = o.5772.

An analysis of a rather large body of data is now in progress. The pilot study on the 40 twins leaves the impression that the slope α of the line is usually less in case the offenses have been recorded than in case the sentences only were available. This holds individually and it may be added that the estimate of α only in a couple of offense records exceeded 1.oo while $\alpha>1$ is considerably more frequent for the sentence records.

7. Discussion. - Pronity and exertion.

Changes in the situation of a human or an animal being may sometimes be described as a sequence of random events, irrespective of whether the changes just happen to it from outside sources, such as cosmic radiation or telephone calls, or they appear to be deliberate actions on the part of the individual.

In studies of accidents the term "proneness" suggests a disposition of the individual to get into trouble. However, when the girls in the ammunition factories (cf. sect. 2) were shown to be subject to accidents at a markedly different scale it might, as far as the observations and the model go, just have been due to differences in the dangerousness of their jobs.

Furthermore, when it is shown that the individual risks for the shunters could be assumed to keep constant throughout 11 years it lays near at hand to assume a certain constancy in the working conditions of each shunter and, accordingly, a certain constancy in his habits, when shunting. But other possibilities may be feasible.

The London busdrivers apparently changed their proneness just after the first year of observation while it stayed constant for the next four years. In this connection it would seem natural to think of some common change of the external conditions, while each driver stuck to his own way of dealing with traffic troubles throughout all the five years.

Altogether it would seem sensible to distinguish between a property of the person, his "pronity", and the "exertion" to which he is exposed through the external conditions, both of them making up his apparent "proneness".

In case the Poisson process holds we may, in analogy to models used in test psychology[1], try to split up the intensity, $\lambda_{\nu i}$, for individual no. ν in situation no. i into two factors:

$$\lambda_{\nu i} = \xi_\nu \varepsilon_i \quad , \tag{7.1}$$

where ξ_ν refers to the individual, ε_i to the situation. As the intensity increases when ξ_ν or ε_i do so, we may take these parameters as formalizing the pronity and the exertion.

For the shunters (7.1) specializes to

$$\lambda_{\nu 1} = \xi_\nu \varepsilon_1, \quad \lambda_{\nu 2} = \xi_\nu \varepsilon_2 \tag{7.2}$$

where ε_1 and ε_2 are the (relative) exertions of the first 6 and the last 5 years, and we have found

$$\varepsilon_1 = \varepsilon_2 \tag{7.3}$$

as a tenable hypothesis, with the conclusion that the exposition of the shunters was largely the same in the two periods.

For the London busdrivers we have already in (3.3) employed the splitting up (7.1) of $\lambda_{\nu i}$ and we only have to recognize the ε's of sect. 3 as the exertion parameters of the three periods considered. In this terminology our results may then be expressed as a constancy of the exertion during the last four years, while the exertion for some reason was some 17% higher in the first year.

The separation (7.1) of the intensity into pronity and exertion became decisive in an analysis of the records of disciplinary measures and other sanctions toward conscripts in the Danish navy.[2] Each conscript serves his term at two or more places which may have rather different traditions for penalizing breakings of military and civil rules. For the appraisal of a particular conscript it therefore ought to be taken into account at which places he served his term. On the other hand, when forming a judgment about a particular place it should be considered which sort of people its officers had to deal with.

Both points of view were allowed for when applying the Poisson process with the intensity (7.1) where we may now think of ξ_ν as the "attitude" of the subscript to the disciplinary rules of the navy, while ε_i stands for the conditions of the place, including possibly both

1) Rasch [10], in particular pp. 16, 41 and 75.

2) Study carried out in the Psychological Service Group of the Defense by Eggert Petersen and the author [9].

how inclined the commissioned and the non-commissioned officers are to using what disciplinary means they have, and how inviting the climate of the place is for breaking the rules. Or to neutralize the language: The pronity-for breaking rules etc. - of the conscript and the exertion of the place.

As the conscripts serve at different sets of places the analysis of the data is too complicated for a presentation on this occasion. But, in fact, it became perfectly possible to estimate the parameters, and to check the model as well.

In the two behavioral experiments of sect. 4 and in the criminal careers of sect. 6 more detailed observations were available, viz. the time intervals between the critical events in question and thus a more thorough probing of our model became feasible. As a result we had to generalize the model to a Poisson process with a variable intensity, at any time depending on the duration until then of the actual state, but still assuming independence of any previous state.

This type of model seemed fairly acceptable to psychologists with whom I have discussed the Poisson process as a possible model for the time aspect of behavior. They found it sensible enough-at least in a first approximation-to assume independence of previous experience when trying to account for a lot of both animal and human behavior. They added, however, that extensions into dynamics would be desirable in order to cover situations where learning, adaptation, etc. are of importance.

But my psychologists were more reserved towards the idea that the probability of a change in behavior should be constant, indifferent to everything. On the contrary, it was suggested, as a rule of some generality, that the longer a behavioral state has been endured, the more unbearable it becomes (e.g. when staring somebody into the eyes). The opposite rule has also fields of application, as in quick adaptations to unpleasant situations (e.g. when getting a finger into water that is a bit too hot).

Adopting, then, the generalized Poisson process as a modified model, the intensity at any time is taken to be a function, $\lambda'(t)$, of time, reckoned, however, from the last critical event, not from the beginning of the record.

In continuance of the previous terminology we may signify $\lambda'(t)$ as the <u>pronity function</u> of the individual under the circumstances considered.

In the three cases in question the pronity function could be described in a very simple way, viz. as proportional to some power of the duration:

$$(7.4) \qquad \lambda(t) = \lambda\alpha t^{\alpha-1},$$

(cf. sect. 5). As regards $\alpha(>0)$ we have to distinguish between three principal cases, cf. fig. 9 of sect. 5:

$\underline{\alpha>1}$, of <u>parabolic pronity</u>, starting from nil and increasing indefinitely with increasing duration;

$\underline{\alpha<1}$, of <u>hyperbolic pronity</u>, infinitely large at $t = 0$, but gradually diminishing towards 0 as time goes on without the change;

$\alpha=1$, of <u>stationary pronity</u>, invariably the same, irrespective of how long the state has so far been endured.

Stationary and parabolic pronities prevailed in the experiments with sticklebacks, while stationary and hyperbolic pronities prevailed in the case of the self-confrontation of the Harvard students.

To this observation it may be briefly commented that the reduction of the pronity in the hyperbolic case <u>may</u> be due to an adaptation process - as getting used to look at oneself - while the increase in the parabolic case <u>may</u> be effected by a rising craving of sexual origin - as it possibly happens to the sticklebacks when turning to the nest.

However, <u>conclusions</u> like that cannot be reached at from the data and the model alone. In particular such interpretations would seem unwarranted in case of the criminals of sect. 6. These we might rather think of as making a living out of their crimes, often with little, sometimes with a larger return - which easily leads to an apparent hyperbolic pronity for offenses. On the other hand, an accumulation of offenses increases the risk of getting caught - and this may very well lead to parabolic pronity for sentences.

In concluding I may add a word to the wise: All models are temporary, no model is true. Not even Newton's classical laws on matter and motion hold against all sorts of data.

Thus it is quite possible that pronities in case of accidents and also in case of sanctions toward conscripts in the Danish navy are in fact far from constant, that future investigations may lead to models like those of sect. 5, say. What we have shown is only that the attempted analysis of the data in hand did not disclose deviations from the constant pronity.

Furthermore the experiments in behavior and the criminal careers lead to a simple type of pronity functions, but it is quite possible that a different function in the long run serves the data better.

And finally the assumed independence of the durations is also open to dispute. As a matter of fact a closer scrutiny of successions of durations as exemplified in fig. 3 has suggested a deepening of the problem in this direction, thus opening up an empirical approach to more dynamic studies in behavior.

Altogether, the results as presented here by no means pretend to be final, and the models may have to be replaced by some other ones tomorrow - that matters little. My main point is that by now it seems worth while to investigate some aspects of behavior in terms of stochastic processes with a continuous time variable, of which class of models the Poisson process is the very simplest.

R e f e r e n c e s.

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Abstract of:

G. Rasch: The Poisson Process as a Model for a Diversity of Behavioral Phenomena.

In a first attempt at studying the time aspect of behavior by means of stochastic processes with a continuous time, the Poisson process and a slight modification of it are employed. The basic assumption of the Poisson process is that at any moment $(t, t+dt)$ there is a probability $\lambda'(t)dt$ of the occurrence of a certain critical event, which is independent of the previous course of the process.

This model forms the basis of the classical studies on accident proneness, but although in previous discussions this concept has always been referred to individuals, the statistical analyses have invariably described variations in populations - and the fit to the negative binomial distribution has even been taken as almost crucial evidence of the presence of accident proneness in a population.

One purpose of the present paper is to test - on published data - the constancy of the accident intensity for each individual. The test rests upon a remarkable property of the Poisson distribution (the rule of conditionality) that eliminates the individual parameters from the test and thus for this purpose abolishes the population. In return, the accident proneness parameter, the intensity, is estimated for each individual - insofar as the model holds.

In the case of 122 South African shunters the test is passed excellently, and the constancy also seems to hold for the last four years of observation of 166 London busdrivers, while the intensity is some 17% higher in the first year.

This observation suggests the accident proneness to be made up of an individual factor, his "pronity", and a factor, the "exertion", signifying the conditions to which all the individuals under consideration are exposed.

These notions, carried over from a recent development in test psychology, seem fruitful in describing the sanctions suffered by conscripts in the Danish navy at different serving-places.

With records of the time points for the critical events in each individual the constant intensity model is such tested that alternatively the intensity is described as a function of the time elapsed since the last critical change of behavior, the "pronity function". In two series of behavioral experiments, one regarding reactions to self confrontation of Harvard Students, the other on sexual behavior of male sticklebacks, the pronity function could be taken as proportional to a power of the duration:

$$\lambda(t) = \lambda\alpha t^{\alpha-1}, \quad (\alpha>0) .$$

For the Harvard Students α was usually found to be ≤ 1, corresponding to an incentive to the antagonistic reaction that immediately is very strong, but by and by fades away. For the sticklebacks α was usually found to be ≥ 1, corresponding to an impulse starting from nil, but increasing indefinitely until the change takes place. In an investigation of criminal careers the same type of function was found, with $a=1$ dominating for criminal acts, and with a quite often ≥ 1 in case of sentencing, the latter often covering an accumulation of offenses.

The desirability of also lessening the assumption of the independence of successive time intervals, thus proceeding to still more dynamic formalizations of behavior, is recognized.

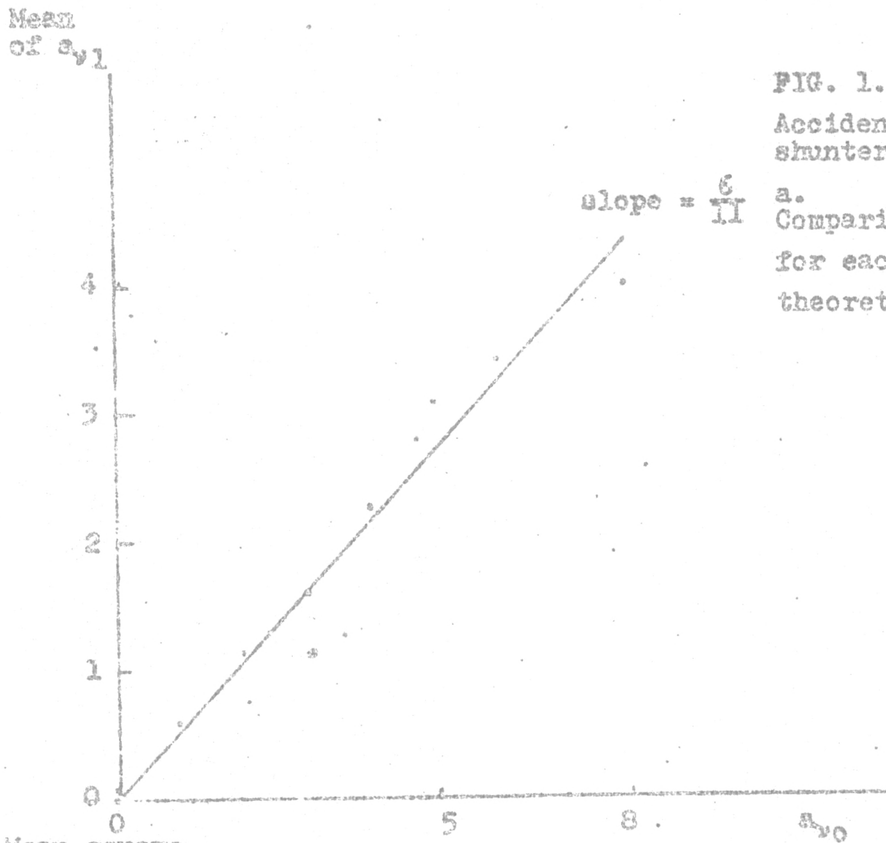


FIG. 1.

Accidents of 122 experienced shunters during 6 + 5 years.

a.
Comparison of mean of $a_{\nu 1}$ for each total $a_{\nu 0}$ with theoretical mean value.

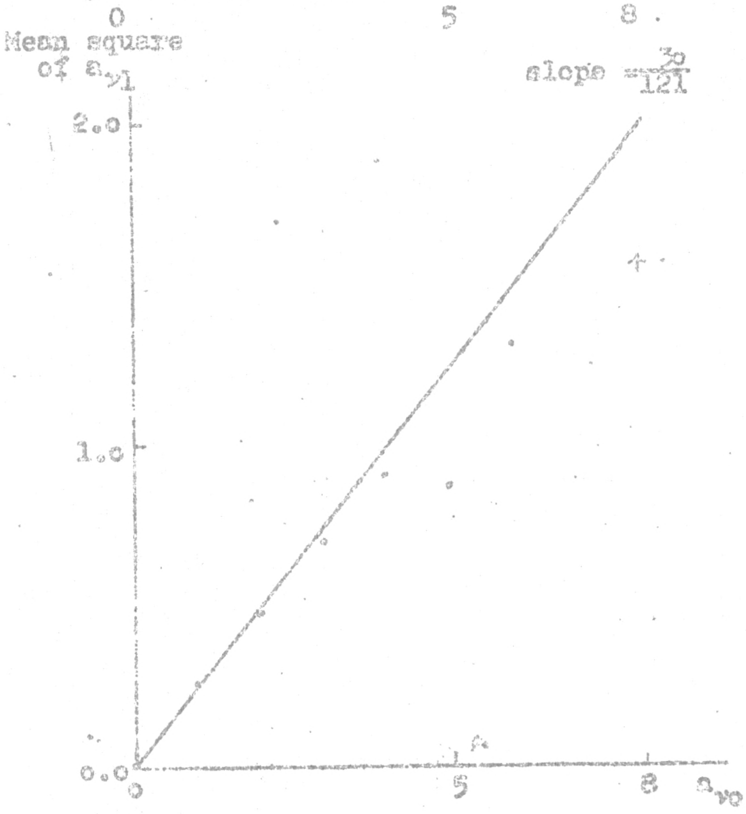


FIG. 1.

b. Comparison of mean square of $a_{\nu 1}$ for each total $a_{\nu 0}$ with theoretical variance.

FIG. 2.

Accidents of 166 London busdrivers in 1 + 4 years.

a. Conditional average numbers of accidents in first year compared to theoretical mean value, assuming constant pronity.

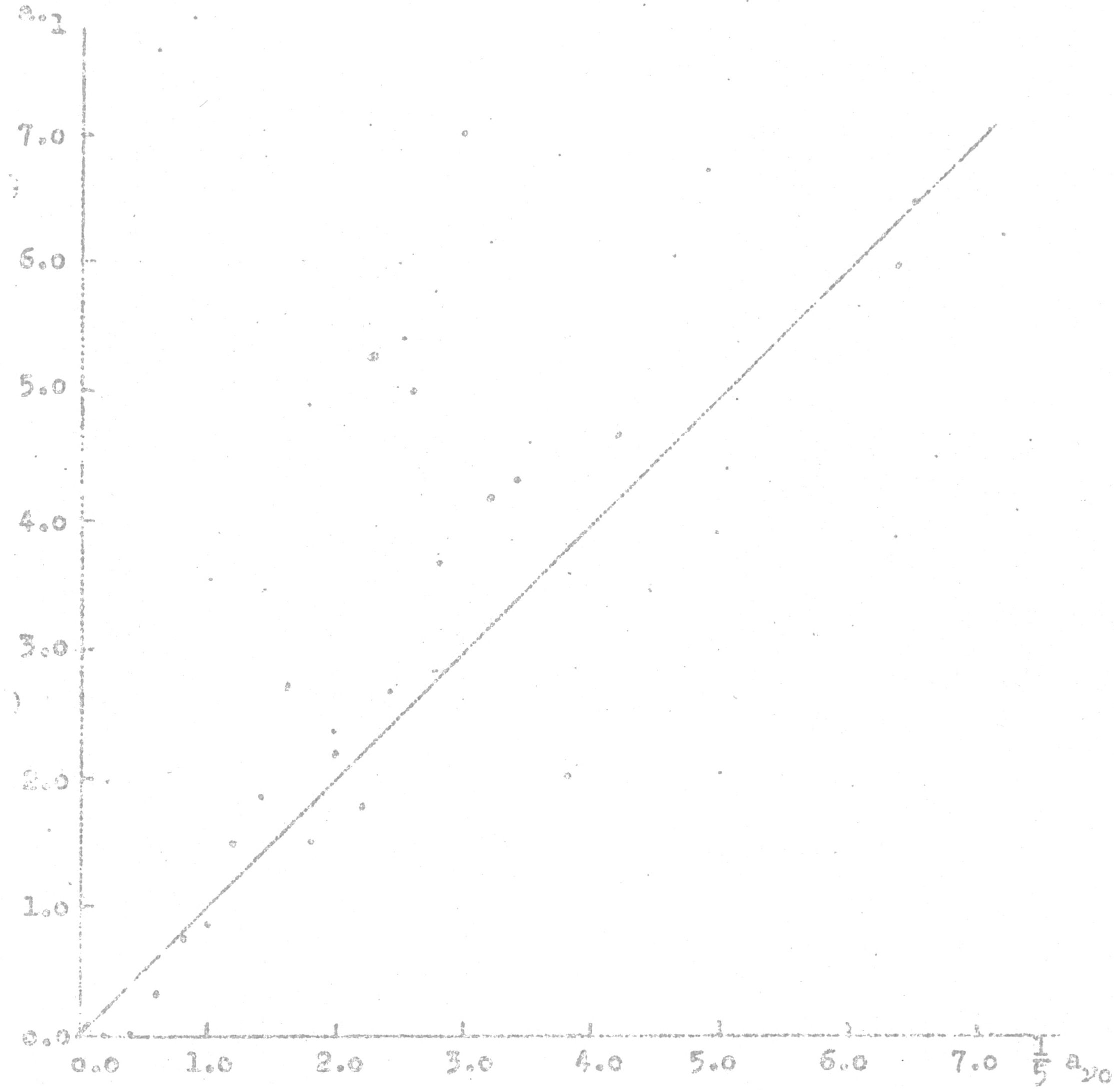
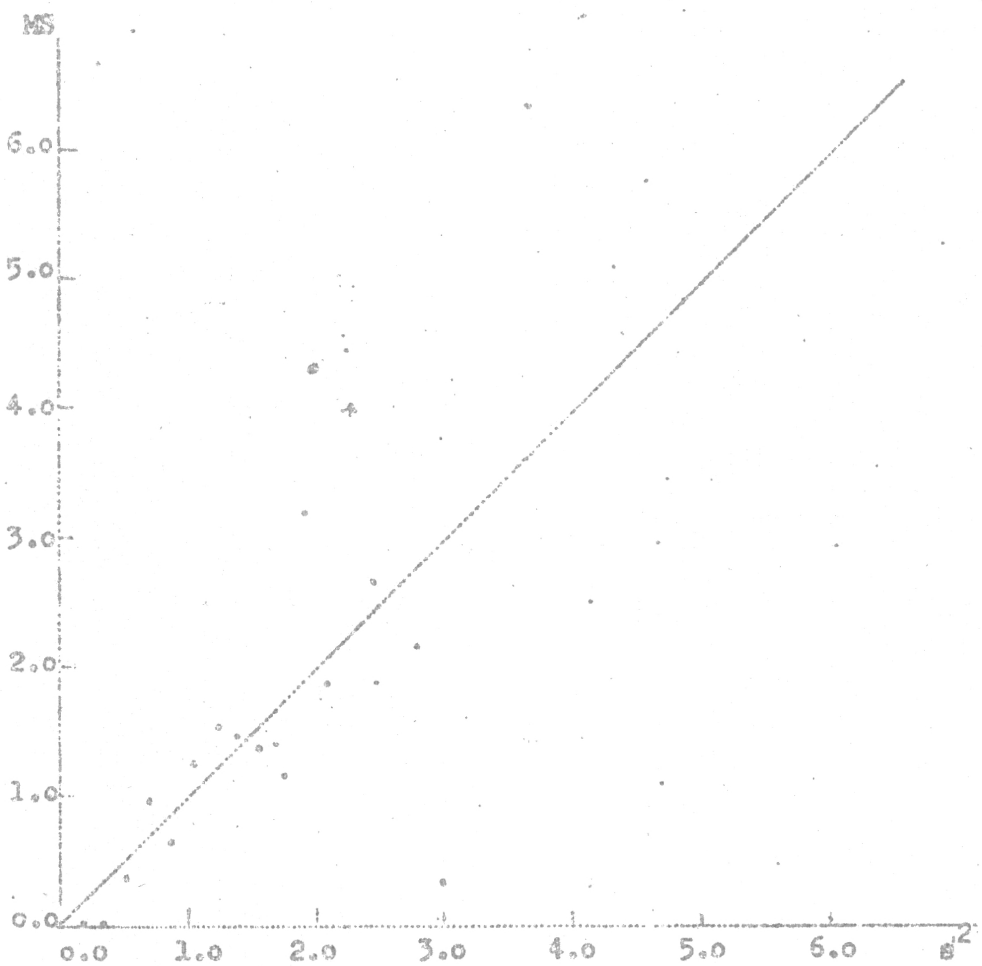


FIG. 2.

b. Conditional mean square of number of accidents in first year compared to variances computed from estimated pronities.



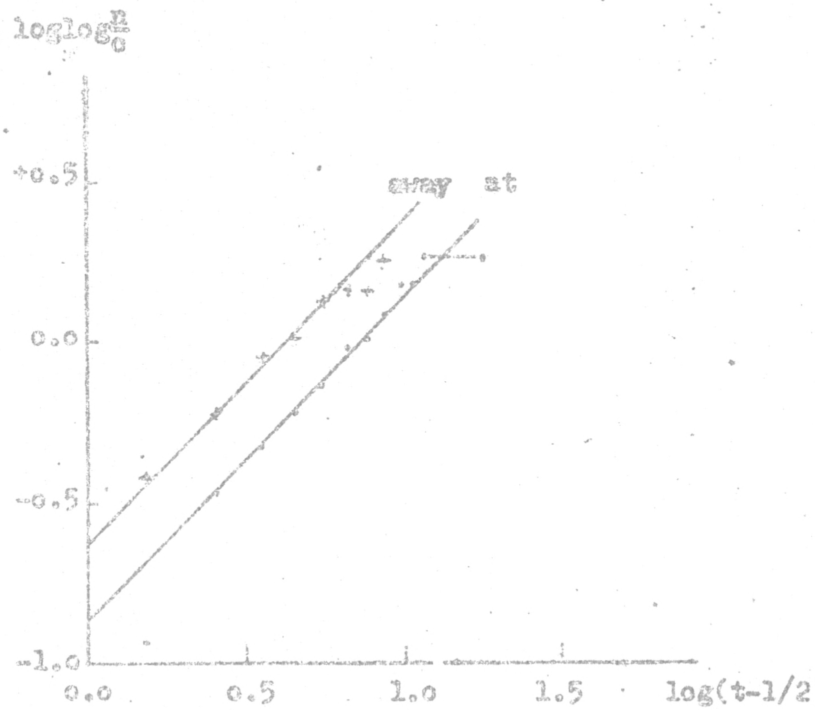


FIG. 4.

Hiber. Second series.
Graphical control on
distribution $\alpha \approx 1.oo$.

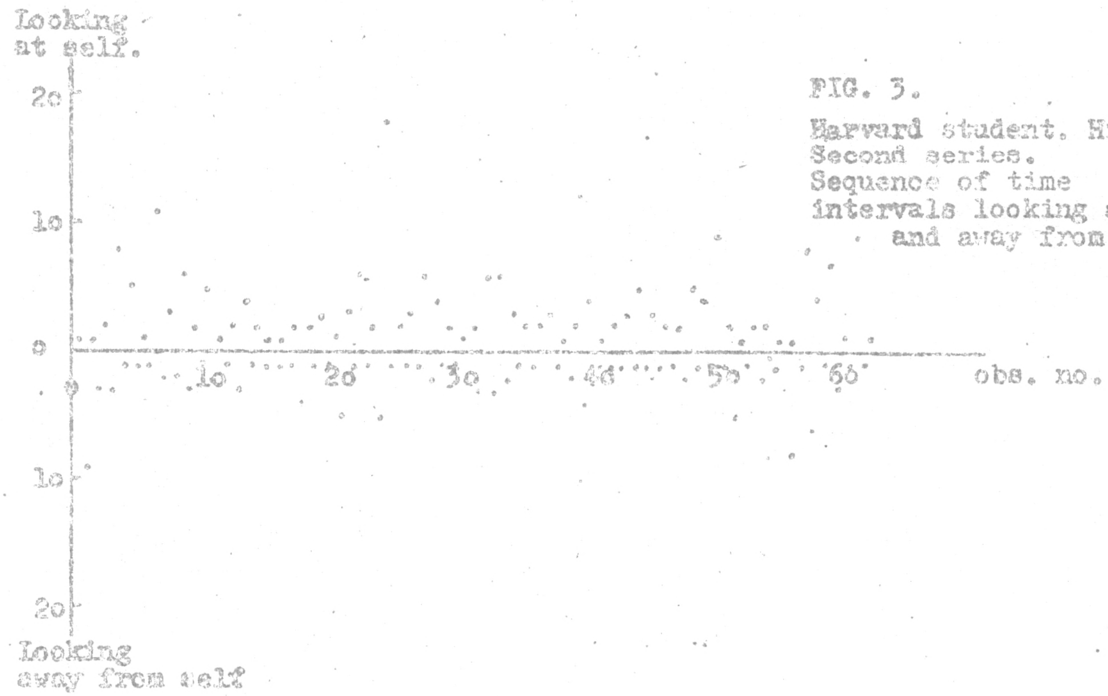


FIG. 3.

Harvard student. Hiber.
Second series.
Sequence of time
intervals looking at
and away from self.

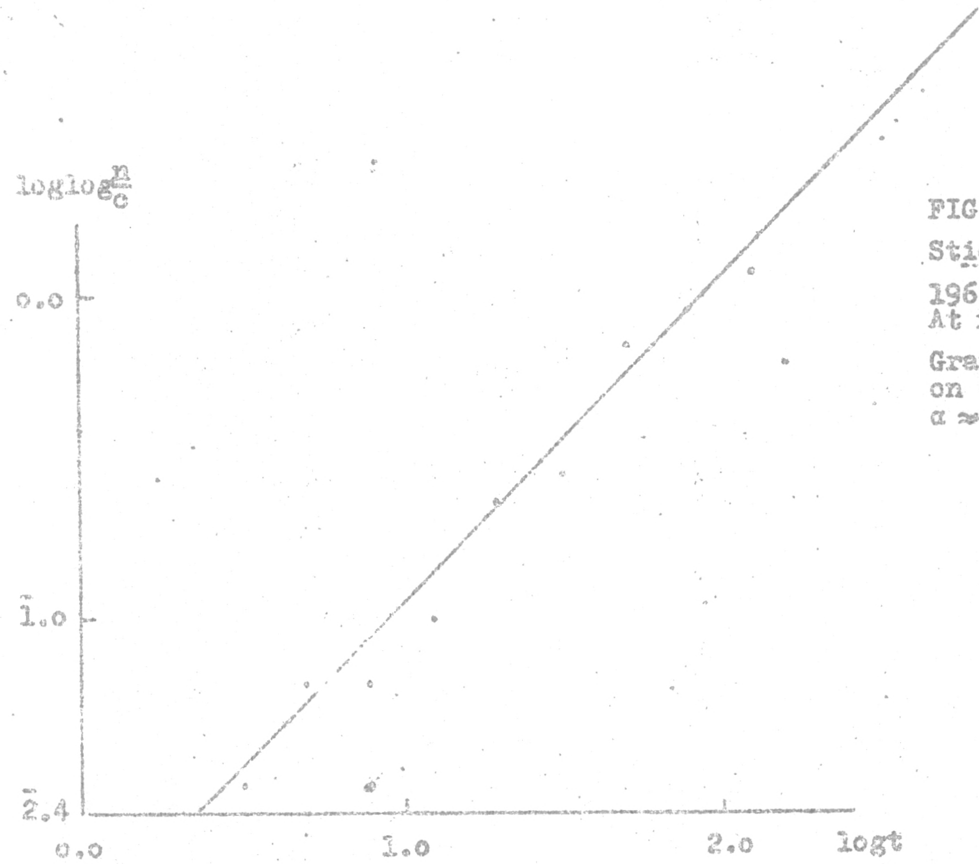


FIG. 5.
Stickleback R 2261.
1962. 12 XII. 13^{38}.
At nest.
Graphical control on distribution.
$\alpha \approx 1.00$.

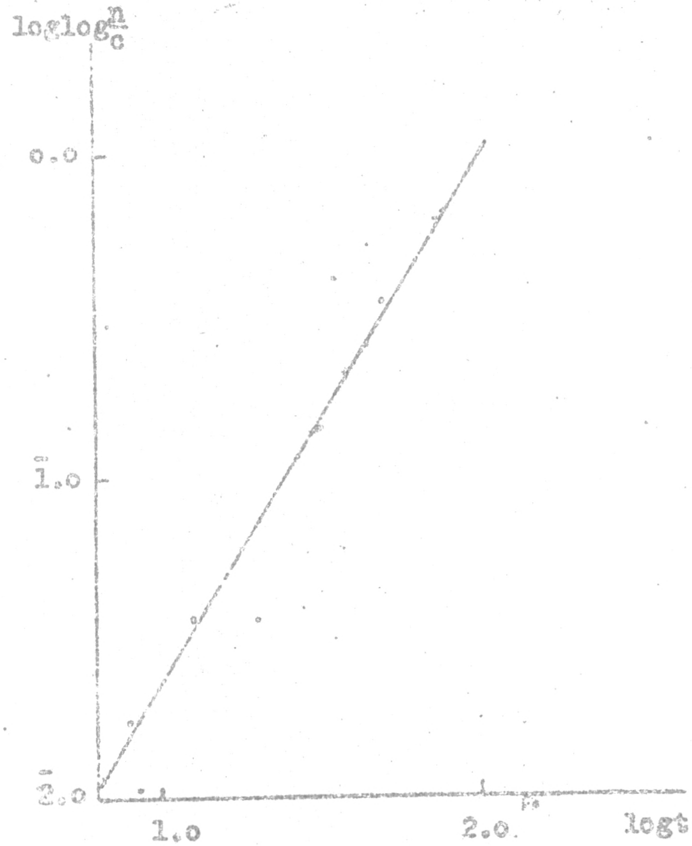


FIG. 6.
Stickleback R 2 A 1o.
1962. 22 XI. 13^{29}.
Away from nest.
$\alpha \approx 1.62$.

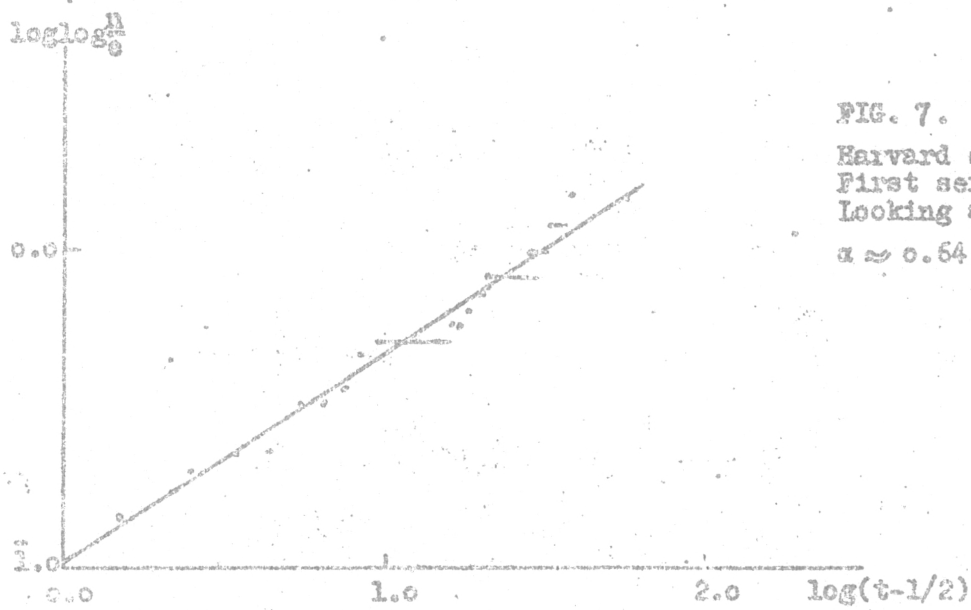


FIG. 7.
Harvard student Gildon.
First series.
Looking at self.
$\alpha \approx 0.64$.

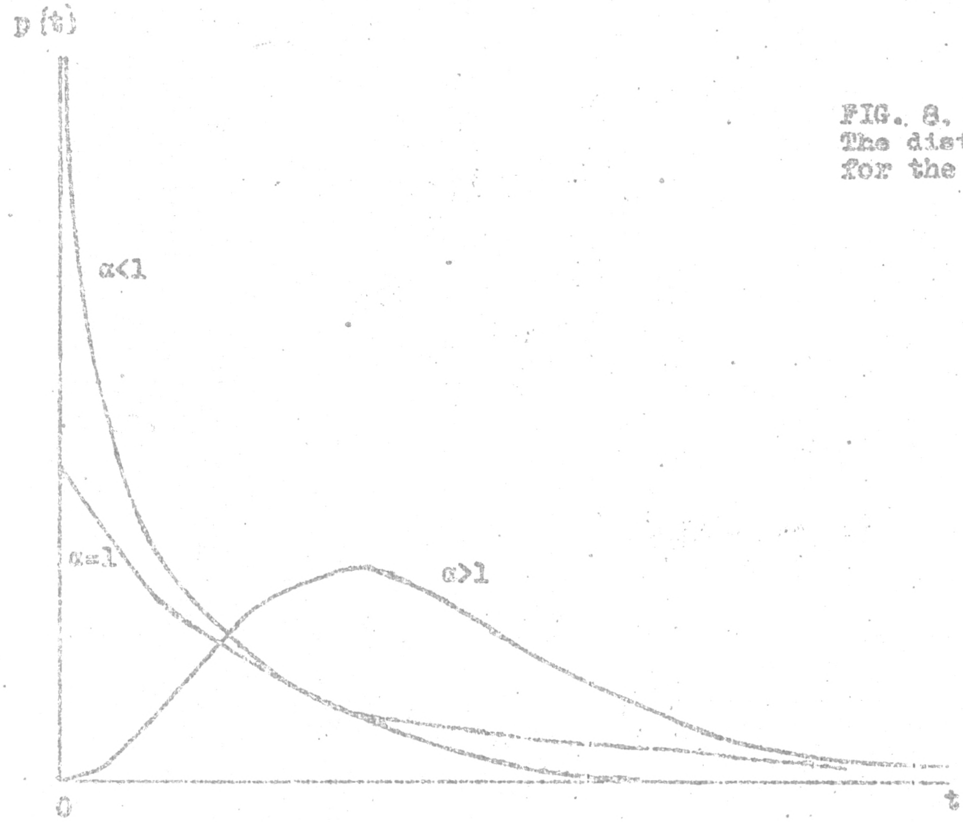


FIG. 8.
The distribution functions
for the duration of a state.

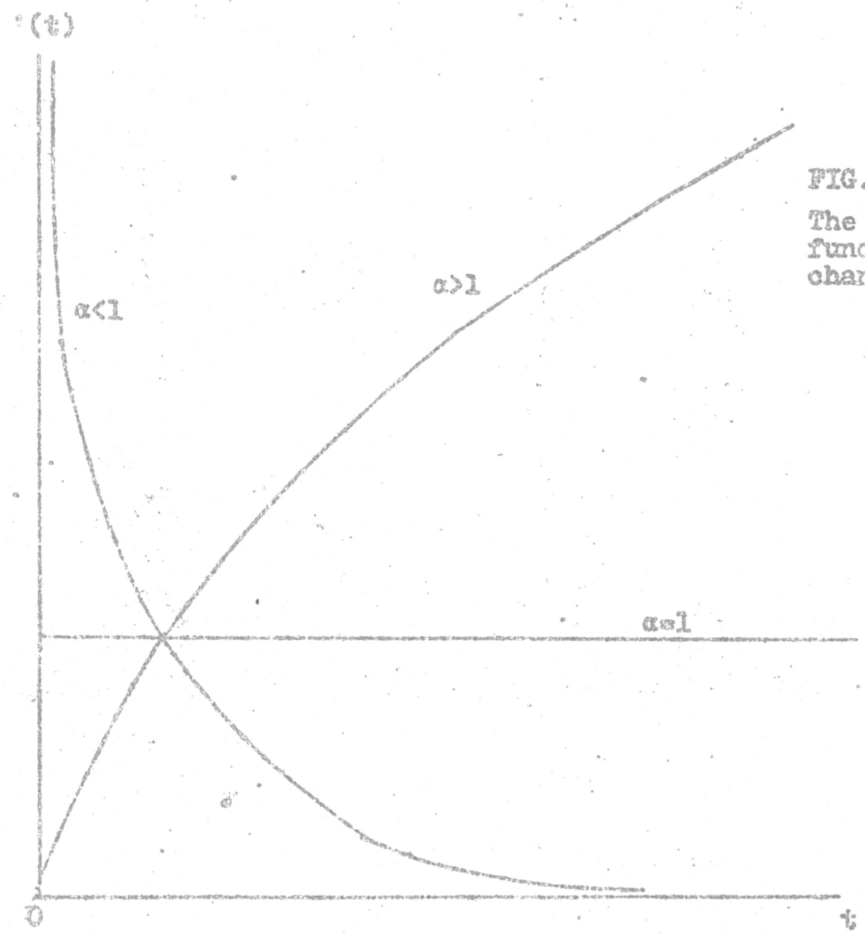


FIG. 9.
The intensity functions for changing a state.

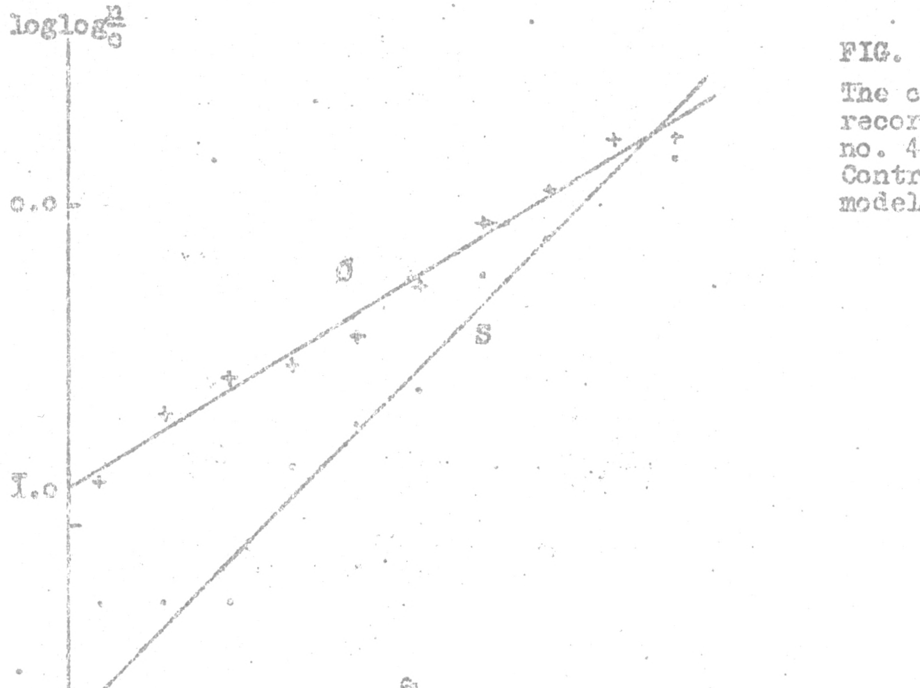


FIG. 10.
The criminal records of twin no. 415a. Control on the model.

$a_{\nu 1}$ \ $a_{\nu 2}$		No. accidents in last 5 years								
		0	1	2	3	4	5	6	7	total
No. accidents in first 6 years	0	21	13	4	2	0	0	0	0	40
	1	18	14	5	1	0	0	0	1	39
	2	8	10	4	3	1	0	0	0	26
	3	2	1	2	2	1	0	0	0	8
	4	1	4	1	0	0	0	0	0	6
	5	0	1	0	1	0	0	0	0	2
	6	0	0	1	0	0	0	0	0	1
	total	50	43	17	9	2	0	0	1	122

Table 1.

Accidents of 122 experienced shunters
during 6 + 5 years. (Arbous and Kerrich).
Data.

$$p\{a_{\nu 1} = 0 \mid a_{\nu 0} = 3\} = \left(\tfrac{5}{11}\right)^3 = 0.094$$

$$p\{a_{\nu 1} = 1 \mid a_{\nu 0} = 3\} = 3 \cdot \tfrac{6}{11} \cdot \left(\tfrac{5}{11}\right)^2 = 0.338$$

$$p\{a_{\nu 1} = 2 \mid a_{\nu 0} = 3\} = 3 \cdot \left(\tfrac{6}{11}\right)^2 \cdot \tfrac{5}{11} = 0.406$$

$$p\{a_{\nu 1} = 3 \mid a_{\nu 0} = 3\} = \left(\tfrac{6}{11}\right)^3 = 0.162$$

Table 2.

Accidents in the first period correspon-
ding to a total of 3 accidents in the whole period.
Terms in the Binomial distribution.

Accidents in T_1	obs	calc.	diff.
0	2	1.8	+0.2
1	5	6.4	-1.4
2	10	7.7	+2.3
3	2	3.1	-1.1
	19		

Table 3.

Accidents in the first period corresponding
to a total of 3 accidents in the whole period.
Comparison of observed and calculated distribution.

$a_{\nu o}$ \ $a_{\nu 1}$		No. accidents in first 6 years 0	1	2	3	4	5	6	7	8	n	Mean	$\mu = a_{\nu o} \cdot \frac{6}{11}$	Mean square	$\sigma^2 = a_{\nu o} \cdot \frac{6}{11} \cdot \frac{5}{11}$
	0	21									21	0.00	0.00	0.00	0.00
	1	13	18								31	0.58	0.55	0.25	0.25
No.	2	4	14	8							26	1.15	1.09	0.46	0.50
acci-	3	2	5	10	2						19	1.63	1.64	0.69	0.74
dents	4	0	1	4	1	1					7	2.28	2.18	0.90	0.99
in	5	0	0	3	2	4	0				9	3.11	2.73	0.86	1.24
11	6	0	0	1	2	1	1	0			5	3.40	3.27	1.30	1.49
years	7	0	0	0	1	0	0	0	0		1	3.00	3.82	-	1.74
	8	0	1	0	0	0	1	1	0	0	3	4.00	4.36	7.00	1.98
											122				

Table 4.

Accidents of 122 experienced shunters during 6 + 5 years. Test of the hypothesis of constant pronity per individual. Mean and mean square of $a_{\nu 1}$ for each total $a_{\nu o}$ compared with mean values and variances of Binomial distributions with parameter = $\frac{6}{11}$.

$a_{\nu o}$ \ $a_{\nu 1}$	0	1	2	3	4	5	6	7	8	n	a_{o1}	Mean a_{o1}	$\frac{a_{\nu o}}{5}$
0	1									1	0	0.00	0.00
1	2									2	0	0.00	0.20
2	3									3	0	0.00	0.40
3	11	2	1							14	4	0.29	0.60
4	9	4	3	1						17	13	0.76	0.80
5	7	11	2	1						21	18	0.86	1.00
6	4	4	7	1	1					17	25	1.47	1.20
7	1	6	3	2	2					14	26	1.86	1.40
8	0	2	5	3	3	1				14	38	2.71	1.60
9	3	3	3	3	0	0				12	18	1.50	1.80
10	1	2	5	4	1	0				13	28	2.15	2.00
11	3	1	3	0	1	1				9	16	1.78	2.20
12		1	2	2	0	1				6	16	2.67	2.40
13			1	0	0	0			1	2	10	5.00	2.60
14			2	1	1	1	1			6	22	3.67	2.80
15			0	0	0	0	0	1		1	7	7.00	3.00
16			1	1	1	2	1	0		6	25	4.17	3.20
17			0		2	1	0	0		3	13	4.33	3.40
18			0			0	0	0		0	0	-	3.60
19			1			0	0	0		1	2	2.00	3.80
20			0			0	0	0		0	0	-	4.00
21			1			1	0	1		1	14	4.67	4.20
---							---			---	---	---	---
32							1			1	6	6.00	6.40
Total										166	301		

$$a_{oo} = \Sigma n a_{\nu o} = 1330.$$

Table 5.

Accidents of 166 London busdrivers in 1 + 4 years.

a. Average numbers of accidents in the 1st year for given total numbers in 5 years.

a_{y0}	SS	f	MS	s^2	SS/a_{y0}
0	0.0	0	–	0.00	
1	0.0	1	0.00	0.18	0.00
2	0.0	2	0.00	0.35	0.00
3	4.9	13	0.38	0.53	1.63
4	15.1	16	0.94	0.70	3.78
5	12.6	20	0.63	0.88	2.52
6	20.2	16	1.26	1.05	3.37
7	19.7	13	1.52	1.23	2.81
8	18.9	13	1.45	1.40	2.36
9	15.0	11	1.37	1.58	1.67
10	13.7	12	1.14	1.75	1.37
11	25.6	8	3.20	1.93	2.32
12	9.3	5	1.86	2.10	0.68
13	18.0	1	18.00	2.28	1.38
14	13.3	5	2.66	2.45	0.95
15	0.0	0	–		0.00
16	10.8	5	2.16	2.80	0.68
17	0.7	2	0.34	2.98	0.04
18	–	–	–		–
19	0.0	0	–		0.00
20	–	–	–		–
21	12.7	2	6.34	3.68	0.66
32	0.0	0	–		0.00
total		145			26.16

Table 5.

Accidents of 166 London busdrivers in 1 + 4 years.

b. Estimation and appraisal of variance of number of accidents in 1st year for given total number in 5 years.

a_{y0} \ a_{y5}	0	1	2	3	4	5	6	7	n	a_{05}	SS/a_{y0}	f
0	1								1	0	–	0
1	2	0							2	0	0.00	1
2	1	2	0						3	2	0.35	2
3	7	5	2	0					14	9	2.40	13
4	6	5	6	0	0				17	17	3.00	16
5	6	10	4	1	0	0			21	21	2.80	20
6	6	8	3	0	0	0	0		17	14	1.42	16
7	1	8	4	0	1	0	0	0	14	20	1.63	13
8	3	5	4	2	0	0	0	0	14	19	1.65	13
9	2	5	3	1	1	0	0	0	12	18	1.67	11
10	2	2	3	5	1	0	0	0	13	27	1.89	12
11	2	0	5	0	3	1	0	0	9	23	2.38	8
12	0	2	1	1	2	0	0	0	6	15	0.79	5
13	1	0	1	0	0	0	0	0	2	2	0.15	1
14	1	1	2	2	0	0	0	0	6	11	0.49	5
15	0	1	0	0	0	0	0	0	1	1	0.00	0
16	0	1	0	2	2	0	0	1	6	22	1.21	5
17	0	1	0	0	1	1	0	0	3	10	0.51	2
18	0	0	0	0	0	0	0	0	0	0	–	–
19	0	0	0	0	1	0	0	0	1	4	0.21	0
20	0	0	0	0	0	0	0	0	0	0	–	–
21	0	0	0	0	2	1	0	0	3	13	0.21	2
.	.	.	.	.	.	.	.	.	.	.	.	
32	0	0	0	0	0	0	0	1	1	7	0.00	0
							a_{00}		=1330.	255	22.76	145

Table 6. Accidents of 166 London busdrivers in 4 + 1 years. Means and sums of squares of number of accidents in last year for given total numbers in 5 years.

Obs. no.	Looking at	Looking away	Obs. no.	Looking at	Looking away	Obs. no.	Looking at	Looking away	Obs. no.	Looking at	Looking away
1	1	9	17	1	1	33	6	3	49	5	1
2	1	3	18	2	4	34	6	2	50	4	1
3	2	3	19	2	1	35	3	1	51	9	1
4	6	1	20	3	1	36	2	1	52	2	5
5	5	1	21	1	5	37	2	3	53	1	1
6	1	1	22	3	1	38	3	1	54	2	2
7	11	2	23	6	1	39	1	2	55	2	1
8	3	2	24	2	5	40	2	4	56	1	8
9	6	3	25	18	1	41	4	2	57	1	1
10	2	1	26	2	1	42	1	1	58	8	6
11	5	2	27	3	1	43	2	1	59	4	1
12	1	1	28	6	2	44	3	1	60	7	3
13	2	3	29	4	1	45	5	1	61	1	1
14	4	1	30	2	1	46	3	1	62	9	1
15	2	1	31	1	2	47	2	1	63	1	-
16	1	1	32	2	3	48	2	2			

Table 7.

Harvard student Hiber. Second series. Sequence of time intervals looking at and away from self.

	Looking at self.					Looking away from self.			
t	$\log(t-\frac{1}{2})$	a	c	$\log\frac{n}{c}$	$\log\log\frac{n}{c}$	a	c	$\log\frac{n}{c}$	$\log\log\frac{n}{c}$
1	-0.301	15	63	0.000	-	36	62	0.000	-
2	0.176	19	48	0.118	-0.928	10	26	0.377	-0.424
3	0.398	8	29	0.337	-0.472	8	16	0.588	-0.231
4	0.544	5	21	0.477	-0.331	2	8	0.889	-0.051
5	0.653	4	16	0.595	-0.225	3	6	1.014	+0.006
6	0.740	5	12	0.720	-0.143	1	3	1.315	0.119
7	0.813	1	7	0.954	-0.021	0	2	1.419	0.152
8	0.875	2	6	1.021	+0.009	1	2	1.419	0.152
9	0.929	2	4	1.197	0.078	1	1	1.792	0.253
10	0.978	0	2	1.498	0.176				
11	1.021	1	2	1.498	0.176				
.	.	.	.	.	.				
18	1.243	1	1	1.799	0.255				

$n = 63$ $n = 62$

Table 8.

Harvard student Hiber. Second series. Distributions of time intervals looking at and away from self. a: absolute frequencies. c: cumulated frequencies.

t	logt	t (grouped)	a	c	$\log\frac{n}{c}$	$\log\log\frac{n}{c}$
161	0.10	1.3	0			
42	.30	2.0	1	15	0.000	–
19	.50	3.2	1	14	.030	2.48
48	.70	5.0	0	13	.062	.79
40	.90	7.9	1	13	.062	.79
62	1.10	12.6	3	12	.097	.99
14	.30	20.0	1	9	.222	I.35
37	.50	31.6	5	8	.273	.44
108	.70	50.1	1	3	.699	.84
5	.90	79.4	1	2	.875	.94
25	2.10	125.9	1	1	1.176	0.07
8	.30	199.5				
2						
41						
20						

$n = 15$

Table 9.

Stickleback R 2261. 1962, 12 XII, 13^{38}. Distribution of time spend at nest. a: frequencies. c: cumulated frequencies.

t	t	logt	t (grouped)	a	c	$\log\frac{n}{c}$	$\log\log\frac{n}{c}$
27	6	0.10	1.3				
62	12	.30	2.0				
65	99	.50	3.2				
97	118	.70	5.0	1	26	0.000	–
42	72	.90	7.9	1	25	.017	2.23
27	47	1.10	12.6	0	24	.035	.54
38	81	.30	20.0	5	24	.035	.54
38	59	.50	31.6	7	19	.136	I.13
21	57	.70	50.1	6	12	.336	.53
98	79	.90	79.4	6	6	.637	.80
104	32	2.10	125.9				
23	42						
	41						
	21						

Table 10.

Stickleback R 2 A 10. 1962, 22 XI; 13^{29}.

Away from nest.

t	$\log(t-\frac{1}{2})$	s	c	$\log\frac{n}{c}$	$\log\log\frac{n}{c}$
	−0.30				
1	+0.18	7	25	0.000	–
2	.40	2	18	.143	$+\bar{1}$.15
3	.54	1	16	.194	.29
4	.65	0	15	.222	.35
5	.74	3	15	.222	.35
6	.81	0	12	.319	.50
7	.88	1	12	.319	.50
8	.93	2	11	.357	.55
9	.98	1	9	.444	.65
10		1	8	.495	.69
.....	.....	.....	.....	.....	.....
	1.22				
17	1.24	1	7	.553	.74
18	1.27	0	7	.553	.74
19	1.29	1	6	.620	.79
20	1.31	1	5	.699	.84
21	1.33	0	5	.699	.84
22	.....	1	4	.796	.90
.....	1.47	.....	.....	.....	.....
30	1.53	1	3	.921	.96
.....	.....	.....	.....	.....	.....
34	1.60	1	2	1.097	0.04
.....	.....	.....	.....	.....	.....
40		1	1	1.398	.15

Table 11.

Harvard student Gildon. First series. Looking at self.

		S.		O.	
log t	t	c	loglog $\frac{n}{c}$	c	loglog $\frac{n}{c}$
-0.1	0.8	24	–	37	–
+0.1	1.3	21	$\bar{2}.76$	27	$\bar{1}$.[illegible]
0.3	2.0	21	$\bar{2}.76$	22	$\bar{1}.35$
0.5	3.2	21	$\bar{2}.76$	19	$\bar{1}.46$
0.7	5.0	17	$\bar{1}.18$	18	$\bar{1}.50$
0.9	7.9	15	$\bar{1}$.[illegible]	15	$\bar{1}.59$
1.1	12.6	13	$\bar{1}.42$	10	$\bar{1}.75$
1.3	20.0	6	$\bar{1}.78$	5	$\bar{1}.94$
1.5	31.6	4	$\bar{1}.89$	3	0.04
1.7	50.1	1	0.14	1	0.20
1.9	79.4	1	0.14	1	0.20
α =			1.00		0.60

Table 12

Evaluation of the criminal records of twin no. 415 a.
t = net periods between successive sentences (S) or offenses (O).
c = cumulated absolute frequencies.