On Objectivity and Specificity of the Probabilistic Basis for Testing.

By

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1. Basic formula for the simple dichotomic model.

One of the situations to be considered in the present paper is the following.

A number of elements, called "objects", characterized by positive scalar parameters, $\xi_1,\dots,\xi_n$, are exposed to a set of other elements, called "agents", also characterized by positive scalar parameters, $\varepsilon_1,\dots,\varepsilon_k$. The "contact" between an object, no. ν, and an agent, no. i, produces with a certain probability, depending upon ξ_ν and ε_i, one out of two possible responses, called 1 and 0.

Denoting the response by $a_{\nu i}$ we then have

$$(1.1) \qquad a_{\nu i} = 1 \text{ or } 0 \quad \text{for } \nu=1,\dots,n,\ i=1,\dots,k$$

For the set of nk responses we assume stochastic independence:

$$(1.2) \qquad p\{((a_{\nu i}))|(\xi_\nu),(\varepsilon_i)\} = \prod_{\nu=1}^{n}\prod_{i=1}^{k} p\{a_{\nu i}|\xi_\nu,\varepsilon_i\}$$

and for each pair (ν,i) the probability is specified as

$$(1.3) \qquad p\{a_{\nu i}|\xi_\nu,\varepsilon_i\} = \frac{(\xi_\nu\varepsilon_i)^{a_{\nu i}}}{1+\xi_\nu\varepsilon_i}.$$

This model was presented in [2] and has been dealt with elsewhere, but for easy reference we shall reproduce the basic steps in the algebraic theory of this "simple dichotomic model".

According to (1.3) the probability (1.2) of any zero-one matrix $((a_{\nu i}))$ is

$$(1.4) \qquad p\{((a_{\nu i}))|(\xi_\nu),(\varepsilon_i)\} = \frac{\prod_{(\nu)} \xi_\nu^{r_\nu} \prod_{(i)} \varepsilon_i^{s_i}}{\gamma((\xi_\nu),(\varepsilon_i))}$$

where r_ν and s_i denote the marginal sums

$$(1.5) \qquad \sum_{(i)} a_{\nu i} = r_\nu\,, \quad \sum_{(\nu)} a_{\nu i} = s_i$$

and where for short

$$(1.6) \qquad \prod_{(\nu)} \prod_{(i)} (1+\xi_\nu \varepsilon_i) = \gamma((\xi_\nu),(\varepsilon_i)) \ .$$

The further algebra may be simplified by using vector-matrix notations like

$$(1.7) \qquad \begin{cases} a_{**} = ((a_{\nu i})) \\ \xi_* = (\xi_1,\ldots,\xi_n), \varepsilon_* = (\varepsilon_1,\ldots,\varepsilon_k) \ , \\ r_* = (r_1,\ldots,r_n), s_* = (s_1,\ldots,s_k) \end{cases}$$

together with the following definition

$$(1.8) \qquad x_*^{a_*} = x_1^{a_1} \ldots x_k^{a_k}$$

of a real vector

$$x_* = (x_1,\ldots,x_k)$$

raised to a power that is an enumerating vector

$a_* = (a_1,\ldots,a_k)$, each a_i being an integer $\lesseqgtr 0$,

and similarly for x_{**} and a_{**} being matrices. The convenience of (1.8) lies partly in the condensation of the formulae and partly in the obvious rule

$$(1.9) \qquad x_*^{a_*} \cdot x_*^{a_*+b_*} \ .$$

The formula (1.4) takes on the form

$$(1.10) \qquad p\{a_{**} | \xi_*, \varepsilon_*\} = \frac{\xi_*^{r_*} \cdot \varepsilon_*^{s_*}}{\gamma\ (\xi_*,\varepsilon_*)} \ .$$

With the symbol

$$(1.11) \qquad \begin{bmatrix} (r_\nu) \\ (s_i) \end{bmatrix} = \begin{bmatrix} r_* \\ s_* \end{bmatrix}$$

for the number of zero-one matrices with the marginal vectors r_* and s_* it follows from (1.10) that

$$(1.12) \qquad p\{r_*, s_* | \xi_*, \varepsilon_*\} = \begin{bmatrix} r_* \\ s_* \end{bmatrix} \frac{\xi_*^{r_*} \; \varepsilon_*^{s_*}}{\gamma(\xi_*, \varepsilon_*)} \quad .$$

Summing over all vectors s_* that are compatible with a given vector r_* the marginal distribution

$$(1.13) \qquad p\{r_* | \xi_*, \varepsilon_*\} = \frac{\gamma(\varepsilon_* | r_*)\xi_*^{r_*}}{\gamma \; (\xi_*, \varepsilon_*)}$$

with

$$(1.14) \qquad \gamma(\varepsilon_* | r_*) = \sum_{(s_*)} \begin{bmatrix} r_* \\ s_* \end{bmatrix} \varepsilon_*^{s_*}$$

obtains, and on dividing that into (1.12) we get the conditional distribution

$$(1.15) \qquad p\{s_* | r_*, \varepsilon_*\} = \begin{bmatrix} r_* \\ s_* \end{bmatrix} \frac{\varepsilon_*^{s_*}}{\gamma(\varepsilon_* | r_*)}$$

Analogously

$$(1.16) \qquad p\{s_* | \xi_*, \varepsilon_*\} = \frac{\gamma(\xi_* | s_*)\varepsilon_*^{s_*}}{\gamma(\xi_*, \varepsilon_*)}$$

with

$$(1.17) \qquad \gamma(\xi_* | s_*) = \sum_{(r_*)} \begin{bmatrix} r_* \\ s_* \end{bmatrix} \xi_*^{r_*}$$

and

$$(1.18) \qquad p\{r_*|s_*,\xi_*\} = \begin{bmatrix} r_* \\ s_* \end{bmatrix} \frac{\xi_*^{r_*}}{\gamma(\xi_*|s_*)} \quad .$$

Finally we divide (1.12) into (1.10) to obtain

$$(1.19) \qquad p\{a_{**}|r_*,s_*\} = \frac{1}{\begin{bmatrix} r_* \\ s_* \end{bmatrix}} \quad .$$

Comments upon appraisal of the parameters. - Specific objectivity.

2. The parameters $\xi_1,\dots,\xi_n$ signify manifestations of a certain property of a set of "objects" which are investigated by means of a set of "agents" characterized by the parameters $\varepsilon_1,\dots,\varepsilon_k$. Thus in principle the ξ's stand for properties of the objects per se, irrespective of which ε_i's might be used for locating them. Therefore they really ought to be appraised without any reference to the ε_i's actually employed for this purpose - just like reading a temperature of an object should give essentially the same result whichever adequate thermometer were used.

Now the joint distribution (1.12) of r_* and s_* depends on both sets of parameters. Therefore an evaluation of $\xi_1,\dots,\xi_n$ based upon this probabilistic statement would in principle be contaminated by the ε_i's . But the conditional distribution (1.18) of r_* for given s_* is independent of ε_* ; therefore an evaluation of ξ_* based upon that statement will be unaffected by which values the elements of ε_* may have. Of course, it depends on s_* , the distribution of which, (1.16), in turn depends on ε_* (and ξ_*) , but this is a different matter: s_* is a known vector, ε_* is not and, in principle, never will become known.

The same argument of course applies to the ε_i's , which by means of (1.15) can be evaluated uninfluenced by the ξ_ν's .

The separation of the parameters achieved by the formulae (1.18) and (1.15) is closely related to the most fundamental requirement in Natural Sciences, in particular in Physics, namely that scientific statements should be <u>objective</u>*).

During centuries philosophers have disagreed about which concept should be attached to the term objectivity, and on this occasion I am not entering upon a discussion of that matter, I only wish to point out that the above mentioned separation exemplifies a type of objectivity which I qualify by the predicate "<u>specific</u>". By which I mean that the statement in question - in the present case one about, say, the ξ-parameters - is not affected by the free variation of contingent factors within the specified frame of reference*). The frame of reference

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*) cf. Hermann Weyl, Philosophy of Mathematics and Natural Science, Princeton University 1947, p. 71.

here being the interaction of objects and agents, the agent parameters are those which are contingent or incidental in statements about the object parameters.

Now a statistician may ask: What is the price of this precious objectivity? Don't we loose some valuable information by insisting upon it?

In the present case the answer is: NO! As a matter of fact, the two conditional distributions (1.15) and (1.18) are equivalent with the joint distribution (1.12).

From the elementary formula

$$(2.1) \qquad p\{r_*, s_* | \xi_*, \varepsilon_*\} = p\{r_* | s_*, \xi_*\} p\{s_* | \xi_*, \varepsilon_*\}$$
$$= p\{s_* | r_*, \varepsilon_*\} p\{r_* | \xi_*, \varepsilon_*\}$$

follows that

$$(2.2) \qquad \sum_{(r_*)} \frac{p\{r_* | s_*, \xi_*\}}{p\{s_* | r_*, \varepsilon_*\}} = \sum_{(r_*)} \frac{p\{r_* | \xi_*, \varepsilon_*\}}{p\{s_* | \xi_*, \varepsilon_*\}} = \frac{1}{p\{s_* | \xi_*, \varepsilon_*\}}$$

which together with $p\{r_* | s_*, \xi_*\}$ produces $p\{r_*, s_* | \xi_*, \varepsilon_*\}$. That the expression thus obtained coincides with (1.12) is easily verified.

To this may be added that when also (1.19) is taken into account, the model itself, as expressed in (1.10), is completely recovered.

3. On the specificity of a model control.

The formula (1.19) tells that the probability of the matrix a_{**}, conditional upon the marginal vectors r_* and s_*, is independent of all of the parameters, thus being a consequence of the structure of the model, irrespective of the values of the parameters. Therefore it would seem a suitable basis for a specifically objective model control, i.e. for testing the validity of the model in a way that is unaffected by the values of the parameters which as regards the structure of the model, are the "contingent factors" within the given framework.

It should be noticed, however, that the derivation of (1.19) only shows that if the model holds then this formula applies, i.e. that (1.19) is a nescessary condition for the model to hold. True enough, according to the preceding section we may, taking (1.2) for granted, work back to (1.3), but only if (1.15) and (1.18) are also taken into account.

The following statement concerns what can be concluded from (1.19) without support from (1.15) and (1.18).

Theorem I. If the elements of a stochastic zero-one matrix

$$(3.1) \qquad a_{**} = ((a_{\nu i})), \quad \nu=1,\ldots,n\ , \quad i=1,\ldots,k$$

are independent:

$$(3.2) \qquad p\{a_{**}\} = \prod_{(\nu)} \prod_{(i)} p\{a_{\nu i}\}\ ,$$

and if the distribution of the matrix, conditional upon the marginal vectors r_* and s_*, defined by (1.5) and (1.7), is given by (1.19) for a particular pair (r_*, s_*) containing no uninformative elements - i.e. each $r_\nu \neq 0$ and k, each $s_i \neq 0$ and n - then two real, positive vectors

$$(3.3) \qquad \xi_* = (\xi_1,\ldots,\xi_n)\ , \quad \varepsilon_* = (\varepsilon_1,\ldots,\varepsilon_k)$$

exist such that (1.3) holds for each pair (ν, i).

Since all of the $a_{\nu i}$'s are either 0 or 1 their probabilities may be represented by

$$(3.4) \qquad p\{a_{\nu i}|\lambda_{\nu i}\} = \frac{\lambda_{\nu i}^{a_{\nu i}}}{1+\lambda_{\nu i}} , \quad \lambda_{\nu i} > 0 .$$

Due to the stochastic independence (3.2) the probability of the matrix a_{**} becomes

$$(3.5) \qquad p\{a_{**}|\lambda_{**}\} = \frac{\lambda_{**}^{a_{**}}}{\prod\limits_{(\nu)} \prod\limits_{(i)} (1+\lambda_{\nu i})} , \quad \lambda_{**} = ((\lambda_{\nu i})) .$$

Summing over a_{**} such that the marginal vectors are kept fixed we get

$$(3.6) \qquad p\{r_*, s_*|\lambda_{**}\} = \frac{\varphi(\lambda_{**}|r_*, s_*)}{\prod\limits_{(\nu)} \prod\limits_{(i)} (1+\lambda_{\nu i})}$$

and in consequence

$$(3.7) \qquad p\{a_{**}|r_*, s_*, \lambda_{**}\} = \frac{\lambda_{**}^{a_{**}}}{\varphi(\lambda_{**}|r_*, s_*)}$$

where

$$(3.8) \qquad \varphi(\lambda_{**}|r_*, s_*) = \sum_{(a_{**})} \lambda_{**}^{a_{**}}$$

with the same summation over a_{**} as above.

Now identification of (3.7) with (1.19) puts such restrictions on the $\lambda_{\nu i}$'s that

$$(3.9) \qquad \lambda_{**}^{a_{**}} = \frac{\varphi(\lambda_{**}|r_*, s_*)}{\begin{bmatrix} r_* \\ s_* \end{bmatrix}}$$

must hold for all a_{**}'s with the marginal vectors r_* and s_* .

Let

$$a_{**} = ((a_{\nu i})) \quad \text{and} \quad a'_{**} = ((a'_{\nu i}))$$

be any two such matrices, then we must have

$$(3.10) \qquad \lambda_{**}^{a_{**}} = \lambda_{**}^{a'_{**}}$$

or, taking logarithms,

$$(3.11) \qquad \sum_{(\nu)} \sum_{(i)} a_{\nu i} \varkappa_{\nu i} = \sum_{(\nu)} \sum_{(i)} a'_{\nu i} \varkappa_{\nu i}$$

where

$$(3.12) \qquad \varkappa_{\nu i} = \log \lambda_{\nu i} \; .$$

According to assumption no r_ν determines all of the elements $a_{\nu 1}, \ldots, a_{\nu k}$, neither does any s_i determine all of the elements $a_{1i}, \ldots, a_{ni}$. Therefore, marking any two object numbers, μ and ν , and any two agent numbers, i and j , it is possible to find two admissible matrices that are identical, except where the rows no. μ and ν intersect the columns no. i and j . In those places we may have the elements

$$(3.13) \qquad \begin{array}{c} \text{in } a_{**} \\ \begin{array}{ccc} & i & j \\ \mu & 1 & 0 \\ \nu & 0 & 1 \end{array} \end{array} \qquad \begin{array}{c} \text{in } a'_{**} \\ \begin{array}{ccc} & i & j \\ \mu & 0 & 1 \\ \nu & 1 & 0 \end{array} \end{array} .$$

Thus (3.11) requires that

$$(3.14) \qquad \varkappa_{\mu i} + \varkappa_{\nu j} = \varkappa_{\mu j} + \varkappa_{\nu i}$$

for any two pairs (μ,ν) and (i,j), even for $\mu=\nu$ and for $i=j$ since (3.14) then is trivial.

Averaging now over μ and j while ν and i are kept fixed we get

(3.14) $\qquad \varkappa_{\nu i} = \varkappa_{\nu .} + \varkappa_{. i} - \varkappa_{..}$

which means that $\lambda_{\nu i}$ factorizes into

(3.15) $\qquad \lambda_{i} = \xi_{\nu}\varepsilon_{i} \quad , \quad \xi_{\nu} = e^{\varkappa_{\nu .}} \quad , \; \varepsilon_{i} = e^{\varkappa_{. i}-\varkappa_{..}} \quad .$

On inserting this into (3.4) we get (1.3), which completes the proof of the theorem *).

With theorem I it becomes clear that, given the independence (3.2), the relation (1.19) is a condition for the validity of the model (1.3) which is both nescessary and sufficient. Therefore I shall characterize (1.19) as <u>a specific basis for testing the structure (1.3) of the model</u>.

At the beginning of this section it was pointed out that (1.19) is a specifically objective basis for the same testing.

These two concepts are fundamentally different.

From (1.19) some more special statements might be derived - e.g. the conditional mean value of the product of all elements of the matrix a_{**} as a function of r_* and s_* . That would still be objective since it is, of course, algebraically independent of the parameters, but being 0 except for all $r_\nu = k$ (and all $s_i = n$), the model cannot be deduced from this value alone. And therefore it is not specific.

On the other hand, one might formulate a statement that is specific, but involves all of the parameters - if nothing else, then the relation (1.4) - and therefore not being objective.

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*) In passing it may be noted that for the extension of (1.3) to to independent binomial distributions

$$p\{a_{\nu i}|m_{\nu i},\xi_\nu,\varepsilon_i\} = \binom{m_{\nu i}}{a_{\nu i}} \frac{(\xi_\nu\varepsilon_i)^{a_{\nu i}}}{(1+\xi_\nu\varepsilon_i)^{m_{\nu i}}} \quad .$$

which may sometimes be useful, both the basic algebra of sect.1 and the inversion theorem I generalize readily.

A probalistic statement that combines the two properties, specificity and specific objectivity, as we have it in (1.19), would seem a most desirable point of departure for entering upon the testing of a hypothesis.

4. On comparing two binomial distributions.

Let a_1 and a_2 be stochastically independent and

$$(4.1a) \qquad p\{a_1|m_1,\zeta_1\} = \binom{m_1}{a_1}\frac{\zeta_1^{a_1}}{(1+\zeta_1)^{m_1}} \quad ,$$

$$(4.1b) \qquad p\{a_2|m_2,\zeta_2\} = \binom{m_2}{a_2}\frac{\zeta_2^{a_2}}{(1+\zeta_2)^{m_2}} \quad ,$$

then

$$(4.2) \qquad p\{a_1,a_2|m_1,m_2,\zeta_1,\zeta_2\} = \binom{m_1}{a_1}\binom{m_2}{a_2}\frac{\zeta_1^{a_1}\zeta_2^{a_2}}{(1+\zeta_1)^{m_1}(1+\zeta_2)^{m_2}} \quad .$$

For the sum

$$(4.3) \qquad c = a_1 + a_2$$

the distribution is

$$(4.4) \qquad p\{c|m_1,m_2,\zeta_1,\zeta_2\} = \frac{\gamma_c(\zeta_1,\zeta_2|m_1,m_2)}{(1+\zeta_1)^{m_1}(1+\zeta_2)^{m_2}}$$

with

$$(4.5) \qquad \gamma_c(\zeta_1,\zeta_2|m_1,m_2) = \sum_{a_1+a_2=c}\binom{m_1}{a_1}\binom{m_2}{a_2}\zeta_1^{a_1}\zeta_2^{a_2}$$

and furthermore

$$(4.6) \qquad p\{a_1,a_2|c,m_1,m_2,\zeta_1,\zeta_2\} = \frac{\binom{m_1}{a_1}\binom{m_2}{a_2}\zeta_1^{a_1}\zeta_2^{a_2}}{\gamma_c(\zeta_1,\zeta_2|m_1,m_2)} \quad .$$

When

$$(4.7) \qquad \zeta_1 = \zeta_2 = \zeta$$

- a hypothesis to be tested - this reduces to

$$(4.8)\qquad \gamma_c(\zeta,\zeta|m_1,m_2) = \binom{m_1+m_2}{c}\zeta^c$$

and

$$(4.9)\qquad p\{a_1,a_2|c,\zeta,m_1,m_2\} = \frac{\binom{m_1}{a_1}\binom{m_2}{a_2}}{\binom{m_1+m_2}{c}}$$

which forms the basis for R.A. Fisher's "exact test" for comparing two relative frequencies [1] .

Clearly (4.9) is independent of the parameter ζ , which according to the hypothesis is common to the two distributions (4.1a) and (4.1b); thus it is specifically objective statement.

But furthermore it is specific for the hypothesis in question:

Theorem IIa. If the independent variables a_1 and a_2 follow the binomial distributions (4.1), and if the conditional distribution (4.6) is given by the right hand term of (4.9), then the two parameters ζ_1 and ζ_2 must coincide.

Theorem IIb. For the coincidence of ζ_1 and ζ_2 in (4.2) it suffices that the distribution (4.6) is algebraically independent of the ratio ζ_1/ζ_2.

To begin with the latter: As (4.5) is homogeneous in ζ_1 and ζ_2 the right hand term of (4.6) depends only on their ratio. And if it is independent of that it must equal its value for $\zeta_1 = \zeta_2$ which according to (4.8) reduces to the right hand term of (4.9).

Identifying next the right hand terms of (4.6) and (4.9) we find that

$$\gamma_c(\zeta_1,\zeta_2|m_1,m_2) = \binom{m_1+m_2}{c}\zeta_1^{a_1}\zeta_2^{a_2}$$

as holding for all a_1 and a_2 with the sum c . (4.7) follows from the identity for $a_1=c$, $a_2=0$ and $a_1=0$, $a_2=c$.

Now and again the adequacy of a test based solely upon (4.9) has been questioned. Realizing, that this condition is both necessary and sufficient for the hypothesis (4.7), provided that we are at all dealing with independent binomially distributed variables, I feel fully satisfied about the adequacy of the basis for Fisher's test.

5. Generalizations.

The results of the preceding section are readily generalized to comparisons of several binomial distributions, of several multinomial distributions, in fact to several "enumerative distributions", i.e. those of the form

$$(5.1a) \qquad p\{a_*|\lambda_*\} = \frac{\alpha_{a_*}\lambda^{a_*}}{\alpha(\lambda_*)}$$

where a_* and λ_* may be vectors:

$$(5.1b) \qquad p\{a_1,\ldots,a_k|\lambda_1,\ldots,\lambda_k\} = \frac{\alpha_{a_1\ldots a_k}\cdot\lambda_1^{a_1}\ldots\lambda_k^{a_k}}{\alpha(\lambda_1,\ldots,\lambda_k)}$$

I may just indicate the steps in comparing two such distributions. For (5.1a) and

$$(5.2) \qquad p\{b_*|\mu_*\} = \frac{\beta_{b_*}\cdot\mu_*^{b_*}}{\beta(\mu_*)}$$

we have, independence being presumed,

$$(5.3) \qquad p\{a_*,b_*|\lambda_*,\mu_*\} = \frac{\alpha_{a_*}\beta_{b_*}\cdot\lambda_*^{a_*}\mu_*^{b_*}}{\alpha(\lambda_*)\beta(\mu_*)}$$

and

$$(5.4) \qquad p\{c_*|\lambda_*,\mu_*\} = \frac{\gamma_{c_*}(\lambda_*,\mu_*)}{\alpha(\lambda_*)\beta(\mu_*)}$$

with

$$(5.6) \qquad c_* = a_* + b_*$$

and

$$(5.7) \qquad \gamma_{c_*}(\lambda_*,\mu_*) = \sum_{a_*+b_*=c_*} \alpha_{a_*}\beta_{b_*}\lambda_*^{a_*}\mu_*^{b_*}$$

Accordingly

$$(5.8) \qquad p\{a_*,b_*|c_*,\lambda_*,\mu_*\} = \frac{\alpha_{a_*}\beta_{b_*}\lambda_*^{a_*}\mu_*^{b_*}}{\gamma_{c_*}(\lambda_*,\mu_*)} \quad ,$$

which for

$$(5.9) \qquad \lambda_* = \mu_*$$

becomes independent of the common parameter:

$$(5.10) \qquad p\{a_*,b_*|c_*\} = \frac{\alpha_{a_*}\beta_{b_*}}{\gamma_{c_*}} \quad ,$$

with

$$(5.11) \qquad \gamma_{c_*} = \sum_{a_*+b_*=c_*} \alpha_{a_*}\beta_{b_*} \quad .$$

(5.10) then is a specifically objective statement, and it is a nescessary condition for (5.9). That it is also sufficient, thus specific for (5.9), is seen by identifying (5.10) with (5.8) which leads to

$$(5.12) \qquad \gamma_{c_*}(\lambda_*,\mu_*) = \gamma_{c_*} \cdot \lambda^{a_*}\mu^{b_*}$$

as holding for any pair (a_*,b_*) satisfying (5.6). It follows that

$$(5.13) \qquad \lambda_*^{a_*}\mu_*^{-a_*} = \prod_{i=1}^{k} \left(\frac{\lambda_i}{\mu_i}\right)^{a_i}$$

must remain constant under the possible variation of a_* which is defined by

$$(5.14) \qquad 0 \leqq a_i \leqq c_i \quad .$$

If for any j with $c_j \geqq 1$ we take $a_j = 1$, all the other a_i's = 0, it is seen that $\lambda_j = \mu_j$. Thus (5.9) must hold, excepting elements with the corresponding $c_j = 0$.

The properties demonstrated in some cases have, of course, to do with the Darmois-Koopman or Exponential family of distributions.

Without going into the general theory of this family I may just indicate how the same train of thought works in the following very simple case of k-dimensional differentiable distribution: With

$$(5.15) \qquad x_* = (x_1, \ldots, x_k), \theta_* = (\theta_1, \ldots, \theta_k)$$

we consider

$$(5.16) \qquad p\{x_* | \theta_*\} = \frac{1}{\gamma(\theta_*)} \cdot e^{\theta_* x_*^*} \alpha(x_*) \quad ,$$

with

$$(5.17) \qquad \gamma(\theta_*) = \int_{-\infty}^{+\infty} e^{\theta_* x_*^*} \alpha(x_*) dx_* \; , \quad dx_* = \prod_{i=1}^{k} dx_i \quad .$$

For two distributions

$$(5.18) \qquad p\{x_{\nu*} | \theta_{\nu*}\} = \frac{1}{\gamma(\theta_{\nu*})} \cdot e^{\theta_{\nu*} x_{\nu*}^*} \alpha_\nu(x_{\nu*}) \; , \quad \nu=1,2$$

of independent variables

$$x_{\nu*} = (x_{\nu 1}, \ldots, x_{\nu k})$$

we have

$$(5.19) \qquad p\{x_{1*}, x_{2*} | \theta_{1*}, \theta_{2*}\} = \frac{\alpha_1(x_{1*})\alpha_2(x_{2*})}{\gamma_1(\theta_{1*})\gamma_2(\theta_{2*})} e^{\theta_{1*} x_{1*}^* + \theta_{2*} x_{2*}^*} \quad .$$

Transforming to x_{1*} and

(5.20) $$z_* = x_{1*}+x_{2*}$$

we get

(5.21) $$p\{x_{1*}, z_* | \theta_{1*}, \theta_{2*}\}$$

$$= \frac{\alpha_1(x_{1*})\alpha_2(z_*-x_{1*})}{\gamma_1(\theta_{1*})\gamma_2(\theta_{2*})} e^{(\theta_{1*}-\theta_{2*})x^*_{1*}+\theta_{2*}z^*_*}$$

and on integration with respect to x_{1*}

(5.22) $$p\{z_* | \theta_{1*}, \theta_{2*}\} = \frac{e^{\theta_{2*}z^*_*}}{\gamma_1(\theta_{1*})\gamma_2(\theta_{2*})}$$

$$\cdot \int_{-\infty}^{+\infty} e^{(\theta_{1_*}-\theta_{2_*})t^*_*}\alpha_1(t_*)\alpha_2(z_*-t_*)dt_*$$

On dividing this into (5.21) the conditional distribution of x_{1*}, given z_*, obtains:

(5.23) $$p\{x_{1*} | z_{1*}, \theta_{1*}, \theta_{2*}\} = \frac{\alpha_1(x_{1*})\alpha_2(z_*-x_{1*})e^{(\theta_{1*}-\theta_{2*})x^*_{1*}}}{\int_{-\infty}^{+\infty} e^{(\theta_{1*}-\theta_{2*})t_*}\alpha_1(t_*)\alpha_2(z_*-t_*)dt_*},$$

which for

(5.24) $$\theta_{1*} = \theta_{2*} = \theta_*$$

simplifies to

$$(5.25) \qquad p\{x_{1*} | z_{1*}, \theta_*\} = \frac{\alpha_1(x_{1*})\alpha_2(z_*-x_{1*})}{\int_{-\infty}^{+\infty} \alpha_1(t_*)\alpha_2(z_*-t_*)dt_*} \quad .$$

This is a probabilistic statement which is independent of the common parameter and therefore specifically objective. And being a consequence of the model (5.19) it is a nescessary condition for the validity of the hypothesis (5.24). However, it is also sufficient and therefore specific for the hypothesis. This is realized by identifying (5.25) with (5.23) , which leads to the identity

$$(5.26) \qquad e^{(\theta_{1*}-\theta_{2*})x_{1*}^*} \int_{-\infty}^{+\infty} \alpha_1(t_*)\alpha_2(z_*-t_*)dt_*$$

$$= \int_{-\infty}^{+\infty} e^{(\theta_{1*}-\theta_{2*})t_*^*} \alpha_1(t_*)\alpha_2(z_*-t_*)dt_* \quad .$$

The right hand term being independent of x_{1*} , the exponential on the left must vanish; thus (5.24) follows.

Literature:

1 R.A.Fisher, Statistical Methods for Research Workers. 12th ed. Oliver and Boyd. 1954. §21.02, p. 96-97.

2 G.Rasch, Probabilistic Models for Some Attainment and Intelligence Tests. Copenhagen 1960. Chapter X, p. 168-182.